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Abstract

摘要

A large class of supergravities in diverse dimensions are surveyed. This includes maximal supergravities, their general gaugings in the framework of embedding tensor formalism, supergravities with less than maximal supersymmetry, their matter couplings and general gaugings. The emphasis is on summarizing their most general form to date, and primarily in their component formulation. A class of exceptional field theories in an extended geometric framework are summarized briefly. For most of the supergravities surveyed, the bosonic part of the Lagrangians and supertransformations up to leading terms in fermions are given.

本文综述了不同维数下的一大类超引力，其中包括最大超引力，介绍了它们在嵌入张量形式框架下的一般规范、超对称性低于最大的超引力，以及它们的物质耦合和一般规范。本文重点总结迄今为止这些理论的最一般形式，且主要围绕其分量表述展开。本文还简要概述了扩展几何框架下的一类例外场论。对于大部分所综述的超引力，本文给出了拉格朗日量的玻色子部分，以及到费米子领头阶的超变换。

Keywords

关键词

Supergravities in diverse dimensions - Gauged and matter coupled supergravities - Embedding tensor formalism - Exceptional field theories - Double field theory

任意维度超引力-定规与物质耦合超引力-嵌入张量形式-例外场论-双重场论

Introduction

引言

After simple supergravity in four dimension was discovered in 1976 [1, 2], supergravities in diverse dimensions were constructed at a rapid pace in subsequent years. The importance of the fact that they appear as low energy limits of string theory has been widely recognized. In a bottom to top approach, supergravities continue to be of great relevance also in studying the effective theories of quantum gravity. Select papers were put together with brief commentaries on supergravities in diverse dimensions (including conformal supergravities), their compactifications, and anomalies in 1989 [3], where an extensive list of references was provided. Since then, the literature on supergravities has expanded greatly, driven considerably by advances made in string theory. Some of the noteworthy developments are as follows: (a) duality symmetries have been explored with remarkable results, (b) exceptional field theories have emerged as manifestly duality invariant formulations of supergravities in diverse dimensions, (c) generalized geometries have been a fertile area of study, (d) the embedding tensor formalism has made it possible to study systematically all possible gaugings of supergravities, (e) progress has been made in the construction of higher derivative extensions of supergravities, (f) anomalies have been probed in more depth, and (g) consistency of higher derivative extended supergravities at different levels has been pursued vigorously in the frame of the swampland program, thereby making progress in addressing the question of uniqueness of the string theory as a UV complete theory of quantum gravity.

> 1976 年四维简单超引力被发现 [1,2]，随后几年，不同维度的超引力被快速构建出来。超引力可作为弦论的低能极限，其重要性已得到广泛认可。在自下而上的研究方法中，超引力在量子引力有效理论的研究中依然至关重要。1989 年已有文献汇编了相关论文，并对不同维度的超引力 (包括共形超引力)、它们的紧致化与反常问题给出了简要评述，同时提供了一份详尽的参考文献列表 [3]。自那时起，在弦论进展的大幅推动下，超引力领域的相关文献数量大幅增长。其中值得关注的部分进展如下:(a) 对偶对称性的研究已取得显著成果，(b) 例外场论已成为不同维度超引力的显式对偶不变表述，(c) 广义几何一直是成果丰富的研究领域，(d) 嵌入张量变格使得系统研究超引力所有可能的规范场成为可能，(e) 超引力高阶导数扩展的构建已取得进展，(f) 反常问题得到了更深入的探究，(g) 在沼泽地计划框架内，不同阶的高阶导数扩展超引力的自洽性得到了大量研究，进而推动了“弦论作为紫外完备的量子引力理论的唯一性”这一问题的研究进展。

Here we shall provide a brief survey of a large class of supergravity theories that have been constructed so far. Similarities between them will become apparent as we span different dimensions and amount of supersymmetry. One might hope to relate them all to string/M theory by several mechanisms that have been discovered. However, as the amount of supersymmetry decreases, more general couplings arise. These may ultimately be related to string/M theory as well by means of yet to be discovered mechanisms. In any event, it is worthwhile to pursue the goal of establishing as many connections as possible among them.

> 本文将对迄今为止构建出的一大类超引力理论做简要综述。当我们涵盖不同维度和不同超对称数量后，这类超引力之间的共性将清晰显现。我们有望通过已发现的多种机制将它们全部联系到弦论/M 理论。然而，随着超对称数量减少，会出现更多类型的广义耦合。这些耦合最终也有可能通过尚未发现的机制与弦论/M 理论建立联系。无论如何，在它们之间建立尽可能多的关联，这一目标仍是值得推进的。

Surveying a large class of supergravities (pure, matter coupled, and gauged) in a limited space is a challenging task. Several omitted topics go beyond the scope of this relatively brief survey. Among the omitted topics are: higher derivative extensions, [1] supergravities on manifold with boundaries, compactifications, consistent truncations, Killing spinors and exact solutions, AdS/CFT correspondence, quantum supergravity and amplitudes. We touch very little the subjects of superspace and anomalies, and we survey briefly the exceptional field theories.

在有限篇幅内对一大类超引力 (纯超引力、物质耦合超引力与规范超引力) 做综述是一项颇具挑战性的工作。若干被省略的主题已经超出了这篇相对简要的综述的范围。被省略的主题包括: 高阶导数扩展、[1] 带边界流形上的超引力、紧致化、自洽截断、基灵旋量与精确解、AdS/CFT 对应、量子超引力与振幅。本文极少涉及超空间与反常问题，仅对例外场论做简要综述。

In this chapter, starting from $11D$ and ending up with $1D$, we survey the supergravities that have been constructed so far at the level of two derivatives. In sections "D = 11", "D = 10", "D = 9", "D = 8", "D = 7", "D = 6", "D = 5", "D = 4", and "D=3", we give the field contents, the bosonic parts of the Lagrangians, and the supertransformations of the fermionic fields in leading order in fermions. The focus on the supertransformations of the fermions is due to their relevance to the definition of Killing spinors. The fact that they are given in leading order in fermions will not be repeated henceforth. Only in $11D$ the full action and supertransformations are given. In section "Supergravities in Extended Geometry Framework", we summarize the $E_{n(n)}$ exceptional field theories for $n = 6, 7, 8$ and $N = 1$ supersymmetric double field theory. The properties of spinors in arbitrary dimensions and the issue of conventions are discussed in Appendix A, the embedding tensor formalism is summarized in Appendix B, and a comprehensive table of symmetric coset spaces arising in pure and matter coupled supergravities is provided in Tables 4, 5, and 6 in Appendix C.

本章我们从 $11D$ 开始，到 $1D$ 结束，综述迄今为止构建出的二阶水平的超引力。在“D = 11”“D = 10”“D = 9”“D = 8”“D = 7”“D = 6”“D = 5”“D = 4”和“D=3”各小节中，我们给出了场内容、拉格朗日量的玻色子部分，以及费米子场领头阶的超变换。聚焦费米子超变换是因为它们和基灵旋子的定义相关。文中不再重复说明超变换都是费米子的领头阶形式。仅在 $11D$ 给出了完整作用量和超变换。在“扩展几何框架下的超引力”小节中，我们总结了 $E_{n(n)}$ 例外场论与 $n = 6, 7, 8$ 和 $N = 1$ 超对称双场论。任意维度旋量的性质与规范约定问题在附录 A 中讨论，嵌入张量变格在附录 B 中总结，纯超引力和物质耦合超引力中出现的对称陪集空间的综合表格在附录 C 的表 4、表 5 和表 6 中给出。

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[1] Higher derivative supergravities in diverse dimensions are reviewed in [4].

[1] 不同维度的高阶导数超引力已在文献 [4] 中综述。

D = 11

Standard 11D Supergravity

标准 11 维超引力

As is well known, eleven is highest dimension in which a supergravity can exist, if fields with spin $s > 2$ are to be avoided. [2] The supergravity multiplet in eleven dimensions is the unique supermultiplet with spin $s \leq 2$ fields, and it consists of the vielbein e_μ^a , the real 3-form potential $A_{\mu\nu\rho}$, and the gravitino ψ_μ , which is a Majorana spinor. The 11D supergravity was constructed in [6]. In the rest of this survey, we give the bosonic parts of the supergravity Lagrangians and the supersymmetry transformations of the fermions only.

However, given the special place $11D$ supergravity occupies in the landscape of supergravities, we shall make an exception and recall the full 2-derivative action, and its full supersymmetry. The full action given is given by [6][3]

众所周知，如果要求避免自旋大于 $s>2$ 的场，那么 11 维是超引力能够存在的最高维度。[2] 11 维超引力多重态是唯一包含自旋不超过 $s\leq 2$ 场的超多重态，它由标架 e_μ^a 、实 3 形式势 $A_{\mu\upsilon\rho}$ 和引力微子 ψ_μ (马约拉纳旋量) 构成。11 维超引力由文献 [6] 构造。在本综述的余下部分，我们仅给出超引力拉格朗日量的玻色部分和费米子的超对称变换。但鉴于 $11D$ 超引力在超引力整体框架中的特殊地位，我们在此破例回顾完整的二导数作用量及其完整超对称变换。完整作用量由 [6][3] 给出

$$\mathcal{L}=eR(\omega)-\frac{1}{48}eF_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}-\frac{1}{144^2}\varepsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\ldots\mu_{11}}$$

$$+e\overline{\psi}_\mu\Gamma^{\mu\upsilon\rho}D_\upsilon\left(\frac{\omega+\widehat{\omega}}{2}\right)\psi_\rho+\frac{1}{192}e\overline{\psi}_{[\lambda}\Gamma^\lambda\Gamma^{\mu\upsilon\rho\sigma}\Gamma^\tau\psi_{\tau]}\left(F_{\mu\upsilon\rho\sigma}+\widehat{F}_{\mu\upsilon\rho\sigma}\right), \tag{1}$$

where

其中

$$\widehat{\omega}_{\mu ab}=\omega^{(0)}_{\mu ab}-\frac{1}{4}\left(\overline{\psi}_\mu\gamma_a\psi_b-\overline{\psi}_\mu\gamma_b\psi_a+\overline{\psi}_a\gamma_\mu\psi_b\right)$$

$$\omega_{\mu ab}=\widehat{\omega}_{\mu ab}-\frac{1}{8}\overline{\psi}_c\gamma_{\mu ab}{}^{cd}\psi_d,$$

$$F_{\mu\upsilon\rho\sigma}=4\partial_{[\mu}A_{\upsilon\rho\sigma]},\ \widehat{F}_{\mu\upsilon\rho\sigma}=F_{\mu\upsilon\rho\sigma}-3\overline{\psi}_{[\mu}\gamma_{\upsilon\rho}\psi_{\sigma]}, \tag{2}$$

and $\omega^{(0)}_{\mu ab}$ is the spin connection without torsion. The local supersymmetry transformations are

且 $\omega^{(0)}_{\mu ab}$ 是无挠自旋联络。局域超对称变换为

$$\delta e_\mu^a=-\frac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu,\ \delta A_{\mu\upsilon\rho}=\frac{3}{2}\bar{\varepsilon}\gamma_{[\mu\upsilon}\psi_{\rho]},$$

$$\delta\psi_\mu=D_\mu(\widehat{\omega})\varepsilon+\frac{1}{288}\left(\gamma_\mu^{\upsilon\rho\sigma\tau}-8\delta_\mu^\upsilon\gamma^{\rho\sigma\tau}\right)\widehat{F}_{\upsilon\rho\sigma\tau}\varepsilon. \tag{3}$$

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[2] We assume (10,1) signature. A locally supersymmetric action in (10,2) dimensions is discussed in [5].

[2] 我们采用 (10,1) 号差。关于 (10,2) 维的局域超对称作用量的讨论见文献 [5]。

[3] In the conventions used here $\{\Gamma_a,\Gamma_b\}=2\eta_{ab}$ with $\eta_{ab}=\mathrm{diag}(-,++\ldots+)$, $\gamma^{a_1\ldots a_{11}}=-\varepsilon^{a_1\ldots a_{11}}$, $\overline{\psi}=\psi^\dagger i\gamma_0$, $D_\mu(\omega)\varepsilon=\partial_\mu\varepsilon+\frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\varepsilon$ and $R=e_a^\mu e_b^\upsilon R_{\mu\upsilon}{}^{ab}$.

[3] 在本文采用的约定中 $\{\Gamma_a,\Gamma_b\}=2\eta_{ab}$ ，满足 $\eta_{ab}=\mathrm{diag}(-,++\ldots+)$, $\gamma^{a_1\ldots a_{11}}=-\varepsilon^{a_1\ldots a_{11}}$, $\overline{\psi}=\psi^\dagger i\gamma_0$, $D_\mu(\omega)\varepsilon=\partial_\mu\varepsilon+\frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\varepsilon$ 和 $R=e_a^\mu e_b^\upsilon R_{\mu\upsilon}{}^{ab}$ 。

The equations of motion also have a rigid scaling symmetry, also known as the trombone symmetry, [4] under which $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ and $A_{\mu\nu\rho} \to \lambda^3 A_{\mu\nu\rho}$. The equations of motion admit $11D$ Minkowski spacetime as a vacuum solution, but not (A)dS spacetime. Furthermore, the action does not admit the introduction of a cosmological constant [9]. A vast literature has accumulated on $11D$ supergravity and especially on its compactifications. For a more detailed review of the $11D$ supergravity itself and the duality symmetries that emerge in its toroidal compacti-fications, see Chap. 40, ”11D Supergravity and Hidden Symmetries”.

运动方程还具有刚性标度对称性,也称为长号对称性,[4] 在该对称性下 $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ 和 $A_{\mu\nu\rho} \to \lambda^3 A_{\mu\nu\rho}$ 。运动方程允许 $11D$ 闵氏时空作为真空解，但不允许 (反) 德西特时空。此外，该作用量无法引入宇宙常数 [9]。目前关于 $11D$ 超引力，尤其是其紧致化的研究文献已有很多。关于 $11D$ 超引力本身及其环面紧致化中涌现的对偶对称性的更详细综述，参见第 40 章“11 维超引力与隐藏对称性”。

Modified 11D Supergravity EOMs

修正的 11 维超引力运动方程

In [10] it was observed that the EOMs of $11D$ supergravity admit a slight modification in which the standard spin connection D can be replaced by a conformal one taking the form $\widehat{D} = D + 2k$, provided that the conformal part of the curvature vanishes, namely $dk = 0$. In simply connected spaces this implies that the one form k is exact and this modification amounts to a field redefinition. However, in non-simply connected spacetimes this modification is non-trivial, as was shown in [11]. This slightly modified $11D$ supergravity is formulated at the level of EOMs only, since it arises in an on-shell Weyl superspace formulation. It was shown in [10] that its dimensional reduction on a circle with a nontrivial Wilson line gives rise to gauged type IIA supergravity in $10D$. This $11D$ theory was called MM theory in [12], where its properties were further discussed. In particular, it was shown that it admits $dS_D \times S^{10-D} \times S^1$ solution for any D , and that while this solution admits a Killing spinor, it is not globally well defined. It has later been shown that the $10D$ EOM’s that are obtained from this theory can also be obtained by a Scherk-Schwarz reduction of $11D$ supergravity on a circle, using the on-shell rigid scale invariance of 11D supergravity equations of motion [13]. [5]

文献 [10] 中发现，$11D$ 超引力的运动方程可以进行微小修正: 标准自旋联络 D 可替换为形式为 $\widehat{D} = D + 2k$ 的共形自旋联络，前提是曲率的共形部分为零，即满足 $dk = 0$ 。在单连通空间中，这意味着一元形式 k 是恰当的，该修正等价于场重定义。然而正如文献 [11] 所示，在非单连通时空中，该修正具有非平庸性。这种经微小修正的 $11D$ 超引力仅在运动方程层面成立，因为它来源于壳外 Weyl 超空间构造。文献 [10] 表明，对其带非平庸威尔逊线的圆周维约化，可得到 $10D$ 中的规范型 IIA 超引力。文献 [12] 将该 $11D$ 理论称为 MM 理论，并进一步讨论了它的性质，尤其指出该理论对任意 D 都容许 $dS_D \times S^{10-D} \times S^1$ 解，且该解存在 Killing 旋量，但并非整体良定义。后续研究表明，该理论得到的 $10D$ 运动方程，也可通过利用 11 维超引力运动方程的壳刚性标度不变性，对 $11D$ 超引力做 Scherk-Schwarz 圆周约化得到 [13]。[5]

Massive 11D Supergravity

有质量 11 维超引力

Motivated by the problem of embedding of type IIA supergravity with Romans mass term into $11D$, which is often referred to as the massive type IIA supergravity, one can construct an action in $11D$ by introducing an auxiliary non-dynamic vector field k^μ with respect to which the Lie derivative of the metric and the 3-form potential, namely $\mathcal{L}_k g_{\mu\nu} = \mathcal{L}_k C_{\mu\nu\rho} = 0$ [14]. The bosonic part of the resulting action is given in [14, eq. (1.15)], where it has been argued to provide a target space background for massive branes. Indeed, its dimensional reduction on a circle produces the type IIA supergravity with the Romans mass deformation [14].

受将包含罗曼斯质量项的 IIA 型超引力 (通常称为有质量 IIA 型超引力) 嵌入 $11D$ 这一问题的启发，我们可以在 $11D$ 中通过引入一个辅助非动力学矢量场 k^μ 来构造作用量，度规和 3 形式势关于该矢量场的李导数为 $\mathcal{L}_k g_{\mu\nu} = \mathcal{L}_k C_{\mu\nu\rho} = 0$ [14]。所得作用量的玻色子部分见文献 [14, 式 (1.15)]，该研究指出它可为有质量膜提供靶空间背景。事实上，它在圆周上的维约化会得到带有罗曼斯质量形变的 IIA 型超引力 [14]。

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[4] This amusing terminology first appeared in [7], where it is motivated by arguing that ”it allows one to scale magnitudes in and out.” As discussed in detail in [8], in a generic two-derivative supergravity theory in D dimensions, the global trombone symmetry is present with $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ and $A_{\mu_1\ldots\mu_p} \to \lambda^p A_{\mu_1\ldots\mu_p}$, scalar fields remaining invariant, and $\psi_\mu \to \lambda^{1/2}\psi_\mu, \chi \to \lambda^{-1/2}\chi$, thus the Lagrangian scaling homogeneously as $\mathcal{L} \to \lambda^{D-2}\mathcal{L}$.

[4] 这个有趣的术语最早出现在文献 [7] 中，其命名动机被表述为“它允许人们按比例放大或缩小量级”。正如文献 [8] 详细讨论的，在 D 维的一般二阶导数超引力理论中，整体长号对称性存在于 $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ 和 $A_{\mu_1\ldots\mu_p} \to \lambda^p A_{\mu_1\ldots\mu_p}$ 下，标量场保持不变，在 $\psi_\mu \to \lambda^{1/2}\psi_\mu, \chi \to \lambda^{-1/2}\chi$ 下，拉格朗日量因此齐次缩放为 $\mathcal{L} \to \lambda^{D-2}\mathcal{L}$ 。

[5] Nonetheless, see [12] for cautionary remarks about the comparison of the mechanism described in [11], with the Scherk-Schwarz reduction described in [13].

[5] 尽管如此，关于文献 [11] 中描述的机制与文献 [13] 中描述的谢尔克-施瓦茨约化的对比，请参阅文献 [12] 中的警示说明。

11D Supergravity with Both Three-Form and Six-Form

同时包含三形式和六形式的 11 维超引力

As in any supergravity theory, one may ask whether a formulation exists in which a set of potential fields may be Hodge-dualized. For early treatment of the role of the six-form potential in $11D$ supergravity, see

[15,16]. The story for dualization of graviton is very complicated and beyond the scope of this survey. Focusing on the three-form potential, one cannot simply add a Lagrange multiplier term $F_4 \wedge dA_6$ and treat F_4 as an independent variable, since it does not appear only through its field strength in the action. Rather, one may look for an action in which both A_3 and A_6 appear. Putting the gravitini aside, the key to this formulation is the duality equation

和所有超引力理论一样，我们自然会问: 是否存在这样一种表述——其中一组势场可以进行霍奇对偶化。关于 $11D$ 超引力中六形式势作用的早期研究，参见文献 [15,16]。引力子的对偶化过程十分复杂，超出了本综述的讨论范围。聚焦于三形式势，我们无法直接添加拉格朗日乘子项 $F_4 \wedge dA_6$ 并将 F_4 视为独立变量，因为它在作用量中并非仅通过其场强出现。相反，我们需要构造一个同时包含 A_3 和 A_6 的作用量。暂不考虑引力微子，这种表述的核心是如下对偶方程

$$\mathcal{O} := F_7 - \star F_4 = 0, \tag{4}$$

where $F_7 = dA_6 - A_3 \wedge dA_3$. There have been several proposals for writing down an action for a p -form potential and its dual $(D-p-2)$ -form in D -dimensions, from which the desired equations motion can be derived. For an extensive list of references on this subject, see for example, see [17, 18] . Here we focus on $11D$ and briefly mention few of the proposals. In one of them, manifest Lorentz covariance is maintained at the cost of introducing a timelike unit vector built out of a real scalar $a(x)$ and a non-polynomial action [19]. In another approach based on [20], an action with a term of the form $\lambda\mathcal{O}^2$ added to $\mathcal{L}_{CJS}$, where λ is a Lagrange multiplier field with suitably symmetries, was considered in [21]. It has been argued in [22], however, while such actions are permissible classically, they have difficulties quantum mechanically if not properly treated. Nonetheless, it may be worth mentioning that, curiously enough a pseudo-Lagrangian of the form

其中 $F_7 = dA_6 - A_3 \wedge dA_3$ 。目前已有多种方案构造 D 维下包含 p 形式势及其对偶 $(D-p-2)$ 形式的作用量，通过这些作用量可以导出期望的运动方程。关于该课题的更多参考文献，例如可参见 [17, 18] 。本文我们聚焦 $11D$ ，简要介绍几种代表性方案。其中一种方案保留了显式洛伦兹协变性，但代价是需要引入由实标量 $a(x)$ 构造的类时单位矢量，且作用量是非多项式的 [19]。另一种方法基于文献 [20]，文献 [21] 考虑了在 $\mathcal{L}_{CJS}$ 中添加形式为 $\lambda\mathcal{O}^2$ 的项的作用量，其中 λ 是具有适当对称性的拉格朗日乘子场。然而文献 [22] 指出，这类作用量虽然经典层面是自洽的，但处理不当会在量子层面出现问题。尽管如此，值得一提的是，一种形式如下的伪拉格朗日量

$$\mathcal{L} = \mathcal{L}_{CJS} + \mathcal{O}_{\mu\nu\rho\sigma}\mathcal{O}^{\mu\nu\rho\sigma}. \tag{5}$$

follows from a consistent truncation of a ”master exceptional field theory” based on E_{11} duality symmetry [23]. The ”pseudo” means that the duality equation is to be imposed by hand after obtaining the equations of motion. Next we mention the approach of [24] where an action is written but at the expense of sacrificing manifest Lorentz covariance. See [25] where this formalism has been applied to $11D$ supergravity. Finally, there exists a proposal [17], which is inspired by string field theory, and has been applied to type IIB supergravity so far, which is manifestly Lorentz covariant, and has finitely many auxiliary fields, and has been asserted that it is amenable to BV quantization.

可以通过对基于 E_{11} 对偶对称性的“主例外场论”做一致截断得到 [23]。这里的“伪”指的是，我们需要在得到运动方程之后手动引入对偶方程。接下来我们介绍文献 [24] 的方案: 该方案写出了作用量，但牺牲了显式洛伦兹协变性。关于该形式体系在 $11D$ 超引力中的应用，参见文献 [25]。最后，文献 [17] 提出了另一种方案，该方案受弦场论启发，目前已应用于 IIB 型超引力，它满足显式洛伦兹协变性，仅包含有限多个辅助场，且被认为适合进行 BV 量子化。

M-Branes and Superspace

M 膜与超空间

$11D$ supergravity has $M2$ -brane [26] and $M5$ -brane solutions [27]. This is consistent with fact that there exist actions for $M2$ -brane and $M5$ -branes which describe their propagation in $11D$ spacetime. In fact their description is best achieved in (11|32) dimensional superspace and relies critically on the presence of a worldvolume local fermionic symmetry known as κ -symmetry. See [28] for the construction of the $M2$ -brane action, and [29] for $M5$ -brane field equations, and [30] for an $M5$ -brane action. It is a remarkable fact that the κ -symmetry of these actions imposes constraints on the target superspace torsion and the super four-form (these are spelled out in [28]) which imply uniquely [10] the standard $11D$ supergravity field equations [6] up to field redefinitions [31,32].

$11D$ 超引力存在 $M2$ 膜 [26] 和 $M5$ 膜解 [27]。这与现有结论一致: $M2$ 膜和 $M5$ 膜存在描述它们在 $11D$ 时空中传播的作用量。实际上，这类膜的最佳描述框架是 (11|32) 维超空间，且高度依赖世界 volume 上一种名为 κ 对称性的局域费米对称性。$M2$ 膜作用量的构造参见 [28]，$M5$ 膜场方程参见 [29]，$M5$ 膜作用量参见 [30]。值得注意的是，这些作用量的 κ 对称性对目标超空间的挠率和超四形式给出了约束 (具体展开见 [28])，在场重定义 [31,32] 下，这些约束唯一推导出标准 $11D$ 超引力场方程 [6][10]。

It is worth noting that $11D$ supergravity also admits the pp-wave solution [33] which may be viewed in as a 1-brane, as well as KK monopole solution which may be considered as a 6-brane [34]. Finally, M9 brane solutions have also been studied [35, 36] , including their relationship to the boundaries of the $M_{10} \times S^1/\mathbb{Z}_2$ that arise in the Horava-Witten construction of $11D$ supergravity on a manifold with boundary [37, 38] .

值得一提的是，$11D$ 超引力也允许 pp 波解 [33]，该解可被视为 1 膜；同时还允许可被视为 6 膜的 KK 磁单极解 [34]。最后，M9 膜解也已有 [35, 36] 相关研究，包括其与 $M_{10} \times S^1/\mathbb{Z}_2$ 边界的关系——这类边界出现在带流形边界 [37, 38] 上 $11D$ 超引力的 Horava-Witten 构造中。

Signature of Spacetime

时空符号差

It was shown in [39] that M-theory in $(1, 10)$ dimensions is linked via dualities to a theory in $(2, 9)$ and $(5, 6)$ dimensions, referred to as $M^\star$ and M' theories, respectively. Various limits of these were shown to give rise to type IIA-like string theories in $(0, 10)$, $(1, 9)$, $(2, 8)$, $(4, 6)$, and $(5, 5)$ dimensions and to type IIB-like string theories in $(1, 9)$, $(1, 9)$, $(3, 7)$, and $(5, 5)$ dimensions.

文献 [39] 已证明，(1,10) 维的 M 理论可通过对偶性分别联系到 (2,9) 维和 (5,6) 维的理论，这两个理论分别被称为 $M^{\star}$ 理论和 M' 理论。研究表明，这些理论的多种极限可以分别产生 $(0,10),(1,9),(2,8),(4,6)$ 维和 $(5,5)$ 维的 IIA 型类弦理论，以及 $(1,9),(1,9),(3,7)$ 维和 $(5,5)$ 维的 IIB 型类弦理论。

D = 10

Type IIA Supergravity and Its Massive Deformations

IIA 型超引力及其有质量形变

The $N=(1,1)$ supergravity in 10D is usually referred to as type IIA or simply IIA supergravity, as it arises in the low energy limit of type IIA string. Its field content is $\left(e_{\mu}^{a}, B_{\mu\nu}, \phi, C_{\mu}, C_{\mu\nu\rho}, \psi_{\mu}, \chi\right)$. The full action and supersymmetry transformations were constructed in [40, 41] . Later, the action was extended by Romans [42] by introduction of a mass parameter m . The bosonic part of this extended action in string frame, and in the notations and conventions of [43], is given by

十维中的 $N=(1,1)$ 超引力通常被称为 IIA 型或简称 IIA 超引力，它源自 IIA 型弦的低能极限。其场内容为 $\left(e_{\mu}^{a}, B_{\mu\nu}, \phi, C_{\mu}, C_{\mu\nu\rho}, \psi_{\mu}, \chi\right)$。完整作用量和超对称变换已在 [40, 41] 中构造完成。随后，Romans 在文献 [42] 中通过引入质量参数 m 对作用量进行了扩展。本文采用文献 [43] 的记号与约定，给出该扩展作用量玻色子部分的弦框架形式如下

$$
\begin{aligned}
S=-\frac{1}{\kappa^{2}} \int d^{10} x \sqrt{-g}\Big\{ & e^{-2\phi}\left[R-4(\partial\phi)^{2}+\frac{1}{2} H^{2}\right]+\frac{1}{2} m^{2}+\frac{1}{2} G_{2} \cdot G_{2} \\
& +\frac{1}{2} G_{4} \cdot G_{4}-\star\left[\frac{1}{2} dC_{3} \wedge dC_{3} \wedge B+\frac{1}{6} m dC_{3} \wedge B \wedge B \wedge B\right. \\
& \left.\left.+\frac{1}{40} m^{2} B \wedge B \wedge B \wedge B \wedge B\right]\right\} \qquad (6)
\end{aligned}
$$

where

其中

$$
G_{2}=dC_{1}+mB,\; G_{4}=dC_{3}+dB \wedge C_{1}+\frac{1}{2} mB \wedge B. \qquad (7)
$$

The fermionic terms and the supertransformations can be found in [43, eqs. (2.33) and (2.34)]. This action is invariant under the rigid scale transformations under which $e^{\phi} \rightarrow \alpha e^{\phi}, C_{3} \rightarrow \alpha C_{3}$ and $B \rightarrow \alpha^{-2} B$ provided that one also scales $m \rightarrow \alpha^{5} m$ [42]. If $m=0$, which is a smooth limit which gives the usual type IIA supergravity, then one must transform instead $C_{1} \rightarrow \alpha^{3} C_{1}$ [40, 41] . Note that this is a genuine (off-shell) symmetry, as opposed to the (on-shell) trombone symmetry. In the presence of the Romans mass parameter m , the theory no longer admits $10D$ Minkowski spacetime as a vacuum solution but it has a domain-wall solution. Furthermore, the two-form potential has eaten the vector field to become massive.

费米子项和超变换可参见文献 [43] 的式 (2.33) 和 (2.34)。只要同时对 $m \to \alpha^5 m$ 做标度变换，该作用量就满足刚性标度变换下的不变性，在此变换下有 $e^\phi \to \alpha e^\phi, C_3 \to \alpha C_3$ 和 $B \to \alpha^{-2} B$ 。若 $m=0$ ，这是一个光滑极限，对应普通 IIA 超引力，此时必须改为变换 $C_1 \to \alpha^3 C_1$ [40, 41] 。请注意这是真正的离壳对称性，不同于 (在壳的) 长号对称性。存在 Romans 质量参数 m 时，该理论不再允许 $10D$ 闵氏时空作为真空解，但存在畴壁解。此外，二形式势“吃掉”矢量场后变成有质量的。

The massive IIA supergravity can be obtained from the massive $11D$ supergrav-ity summarized earlier by dimensional reduction on a circle [14]. As mentioned earlier, there also exists another massive deformation of type IIA supergravity, which can be obtained from $11D$ supergravity by Scherk-Schwarz reduction on a circle, in which the $11D$ on-shell trombone symmetry is employed, thereby giving rise to a gauged theory in $10D$. In this gauged type IIA theory, the vector eats a scalar, and the three-form eats the two-form to become massive. The bosonic field equations of the gauged type IIA theory are provided in [13, Sec. 5], and they clearly exhibit the Stuckelberg symmetries associated with the Higgs mechanism responsible for the 1-form and 3-form fields becoming massive. See also [44] where the generalized reduction of the $11D$ supertransformations to $10D$, as well as vacuum solutions, are given. The resulting theory can also be viewed as type IIA supergravity in which a combination of the on-shell trombone symmetry (leaving the $10D$ scalars invariant) and an off-shell $GL(1)$ symmetry (leaving the of the $10D$ metric invariant) of type IIA supergravity is gauged. This phenomenon is explained in detail in [8] where a systematic framework for the classification and construction of these theories is set up, using the embedding tensor formalism (see Appendix B here). This theory has no D8 brane solution but it admits a supersymmetric and non-static cosmological solution [11], as well as nonsupersymmetric de Sitter solution [12,13].

有质量 IIA 超引力可以通过前文总结的有质量 $11D$ 超引力做圆周维约化得到 [14]。如前所述，IIA 超引力还存在另一种有质量形变，它可以通过 Scherk-Schwarz 约化从 $11D$ 超引力得到: 在圆周约化中利用了 $11D$ 的在壳长号对称性，最终在 $10D$ 中得到一个规范理论。在这个规范 IIA 理论中，矢量吃掉一个标量，三形式吃掉二形式后变成有质量的。规范 IIA 理论的玻色子场方程已在文献 [13] 第 5 节给出，其中清楚展示了与 1 形式场、3 形式场获得质量的希格斯机制相关的施图克尔贝格对称性。另可参见文献 [44]，其中给出了 $11D$ 超变换到 $10D$ 的广义约化结果，以及真空解。得到的理论也可以视为这样一种 IIA 超引力: 它对 IIA 超引力的一个组合对称性做了规范，该组合由保持 $10D$ 标量不变的在壳长号对称性和保持 $10D$ 度规不变的离壳 $GL(1)$ 对称性构成。这一现象在文献 [8] 中有详细解释，该文献利用嵌入张量形式 (见本文附录 B) 建立了对这些理论进行分类和构造的系统框架。该理论不存在 D8 膜解，但允许超对称非静态宇宙学解 [11]，也允许非超对称德西特解 [12,13]。

The Democratic Formulation

共形表述

One may consider the dualization of the RR fields C_3, C_1 in (6) to obtain an action in which the dual potentials A_5, A_7 appear instead. To do this, the field C_3 needs to be redefined so that all RR fields appear only under derivatives in the action. This is possible by defining $A_3 = C_3 - C_1 \wedge B$. However, adding the Lagrange multiplier terms $\int (dC_1 \wedge dA_7 - dA_3 \wedge dA_5)$, and treating G_2 and G_4 as integration variables, leads to coupled and complicated nonlinear equations in terms of the dual field strengths. One can formally solve these equations at the expense of having inverses of B dependent functionals, and it does not seem to be illuminating. However, it is possible to write down a simple pseudo-action in which both the RR fields and their duals appear. This is referred to as the democratic formulation, and the bosonic part of the action takes the form [43]

我们可以对式 (6) 中的 RR 场 C_3, C_1 做对偶化, 得到一个以对偶势 A_5, A_7 替代原场的作用量。为此, 需要对场 C_3 重新定义, 使得所有 RR 场在作用量中仅以导数形式出现, 这可以通过定义 $A_3 = C_3 - C_1 \wedge B$ 实现。但如果加入拉格朗日乘子项 $\int (dC_1 \wedge dA_7 - dA_3 \wedge dA_5)$，并将 G_2 和 G_4 视为积分变量，最终会得到对偶场强下耦合的复杂非线性方程。我们可以形式上求解这些方程，但结果会得到依赖于 B 逆的泛函，并不具备启发性。不过我们可以写出一个简洁的伪作用量，其中同时包含 RR 场及其对偶场，这就是所谓的共形表述，该作用量的玻色子部分形式如下 [43]

$$S = -\frac{1}{\kappa^2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2} H^2 \right] + \frac{1}{2} \sum_{n=0}^{5} G_{2n} \cdot G_{2n} \right\}, \tag{8}$$

where

其中

$$G = dC - dB \wedge C + G_0 e^B, \tag{9}$$

in which formal sums $C = C_1 + C_3 + \ldots + C_9$ and $G = G_0 + G_2 + \ldots + G_{10}$ are understood. Note that there are no explicit Chern-Simons term in the action. This is a pseudo-action because the duality equations must be put in by hand, after varying action. These duality equations take the form $G_{2n} + \Psi_{2n} = (-1)^{[n]} \star G_{(10-2n)}$, where $\Psi_{(2n)}$ are certain fermionic bilinear terms that can be found in [43], where the fermionic terms in the action and the supertransformations that leave the action (8) invariant are also given.

式中的形式和 $C = C_1 + C_3 + \ldots + C_9$ 与 $G = G_0 + G_2 + \ldots + G_{10}$ 是约定写法。注意该作用量中没有显式的陈-西蒙斯项。它是伪作用量，因为对偶化方程需要在对作用量变分后手动引入，这些对偶方程形式为 $G_{2n} + \Psi_{2n} = (-1)^{[n]} \star G_{(10-2n)}$，其中 $\Psi_{(2n)}$ 是特定的费米子双线性项，可在文献 [43] 中找到，该文献同时也给出了作用量中的费米子项，以及使作用量 (8) 不变的超变换。

Superspace

超空间

Type IIA supergravities provided consistent target spaces for super D -branes, as well as Type IIA string. The requirement of local κ -symmetry of their worldvol-ume description imposes constraints on target superspace torsion and appropriate superforms. Some of these constraints are conventional and may differ as they amount to field redefinitions. Up to such field redefinitions, superspace constraints that describe Type IIA supergravity are given in [45-48]. [6]

IIA 型超引力为超 D -膜以及 IIA 弦提供了自洽的靶空间。对其世界体描述要求局域 κ 对称性，这给靶超空间的挠率和恰当的超形式施加了约束。其中部分约束是约定性的，由于等价于场重定义，不同方案下约束可以不同。在场重定义的意义下，描述 IIA 型超引力的超空间约束已给出在文献 [45-48] 中。[6]

Type IIB Supergravity

IIB 型超引力

The $N = (2,0)$ supergravity in $10D$ is usually referred to as type IIB supergravity as it arises in the low energy limit of type IIB string theory. The multiplet is $\left(e_\mu^a, B_{\mu\nu}, \phi, C, C_{\mu\nu}, C_{\mu\nu\rho\sigma}, \psi_\mu, \chi\right)$ where the C -fields are real 0,2,4 form fields and the fermions are Weyl. On-shell the field strength of C_4 is self-dual, and this is an obstacle for writing a (standard) manifestly Lorentz invariant action. On the other hand, manifestly Lorentz covariant field equations of motion have been constructed [49, 50]. A pseudo-action can also be constructed in the sense that the correct equations of motion are obtained by using self-duality equation by hand after the variation of the action. The bosonic part of such an action, in the string frame, is given by [51,52]

$N = (2,0)$ 超引力在 $10D$ 中通常被称为 IIB 型超引力，因为它出现在 IIB 型弦论的低能极限下。其超多重态为 $\left(e_\mu^a, B_{\mu\nu}, \phi, C, C_{\mu\nu}, C_{\mu\nu\rho\sigma}, \psi_\mu, \chi\right)$，其中 C 场是实 0 型、2 型、4 型形式场，费米子为外尔费米子。在壳条件下，C_4 的场强是自对偶的，这是构造标准的、明显洛伦兹不变作用量的障碍。另一方面，明显洛伦兹协变的场运动方程已经被构造出来 [49, 50]。我们也可以构造伪作用量: 对作用量变分后手动代入自对偶方程，即可得到正确的运动方程。该伪作用量的玻色子部分在弦框架下的表达式为 [51, 52]

$$
\begin{aligned}
S = \frac{1}{2}\int d^{10}x\sqrt{-g}\Big\{&e^{-2\phi}\left[-R + 4\partial_\mu\phi\partial^\mu\phi - \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \\
&-\frac{1}{2}\partial_\mu C\partial^\mu C - \frac{3}{4}\left(F_{\mu\nu\rho} - CH_{\mu\nu\rho}\right)\left(F^{\mu\nu\rho} - CH^{\mu\nu\rho}\right) \\
&-\frac{5}{6}F_{\mu_1\ldots\mu_5}F^{\mu_1\cdots\mu_5} - \frac{1}{48}\varepsilon^{\mu_1\cdots\mu_{10}}H_{\mu_1\ldots\mu_3}F_{\mu_4\ldots\mu_6}C_{\mu_7\ldots\mu_{10}}\Big\},
\end{aligned}
\tag{10}
$$

where

其中

$$
H = dB,\ F_3 = dC_2,\ F_5 = dA_4 + \frac{3}{4}\left(B\wedge F_3 - C_2\wedge H\right). \tag{11}
$$

---

[6] The relation between those of [47] to others remains to be shown.

[6] 文献 [47] 中结果与其他结果的关系仍有待证明。

The fermionic terms to be added to this pseudo-action and the attendant supersym-metry transformations remain to be spelled out. See, however, [53,54] for more information. The $SL(2,\mathbb{R})$ symmetry of the action above is not manifest but it is in a convenient form for passing to the democratic formulation. Indeed, a democratic formulation of this pseudo-action in which duals of RR fields are introduced is also available and it takes same form as in Eqs. (8) and (9), now with $n = 1/2, \dots, 9/2$, and $C = C_1 + C_2 + \dots + C_8$ and $G = G_1 + \dots + G_9$. In this case too the fermion terms and supertransformations can be found in [43].

需要添加到该伪作用量中的费米子项，以及对应的超对称变换仍有待明确给出，更多信息可参见 [53, 54]。上述作用量的 $SL(2,\mathbb{R})$ 对称性并不明显，但它便于推广到民主表述。实际上，该伪作用量已经有了引入 RR 场对偶形式的民主表述，其形式与 (8)、(9) 式一致，此时对应参数为 $n = 1/2, \dots, 9/2$ 、$C = C_1 + C_2 + \dots + C_8$ 和 $G = G_1 + \dots + G_9$ 。这种情况下的费米子项和超变换也可在文献 [43] 中找到。

It is often very useful to express the type IIB supergravity equations in manifestly in $SL(2,\mathbb{R})$ invariant form. The bosonic part of the type IIB (pseudo)action in Einstein frame is given by [55]

将 IIB 型超引力方程表示为明显满足 $SL(2,\mathbb{R})$ 不变性的形式通常非常有用。爱因斯坦框架下 IIB 型 (伪) 作用量的玻色子部分表达式为 [55]

$$S^{(0)}_{IIB} = \frac{1}{2\kappa_{10}^2}\int\left(R - 2P_\mu\bar{P}^\mu - \frac{|G_3|^2}{2\cdot 3!} - \frac{|F_5|^2}{4\cdot 5!}\right)\star\mathbb{1}$$

$$+\frac{1}{8i\kappa_{10}^2}\int C_4\wedge G_3\wedge\bar{G}_3 \tag{12}$$

where

其中

$$\tau = C_0 + ie^{-\phi},\ P_\mu = \frac{i}{2}(\tau_2)^{-1}\nabla_\mu\tau,\ G_3 = (\tau_2)^{-1/2}(F_3 - \tau H_3),$$

$$H_3 = dB_2,\ F_3 = dC_2,\ F_5 = dC_4 - \frac{1}{2}H_3\wedge C_2 + \frac{1}{2}F_3\wedge B_2, \tag{13}$$

and $\tau = \tau_1 + i\tau_2$. The supertransformations of the fermionic fields are [56]

且 $\tau = \tau_1 + i\tau_2$ 。费米子场的超变换表达式为 [56]

$$\delta\psi_\mu = D_\mu\varepsilon + \frac{1}{480}i\gamma^{\mu_1\cdots\mu_5}\gamma_\mu F_{\mu_1\dots\mu_5} + \frac{1}{96}\left({\gamma_\mu}^{\nu\rho\sigma} - 9\delta^\nu_\mu\gamma^{\rho\sigma}\right)G_{\nu\rho\sigma},$$

$$\delta\chi = i\gamma^\mu\varepsilon^\star P_\mu \tag{14}$$

As is well known, the gravitational anomalies cancel exactly in this theory [57]. As for the composite local $U(1)$ symmetry, it was noted in [58] that it is anomalous. The anomaly vanishes upon the use of Maurer-Cartan equation though, and therefore there are no anomalies in the fundamental theory but the composite $U(1)$ vector field cannot become dynamical [59].

众所周知，该理论的引力反常可以完全抵消 [57]。对于复合局域 $U(1)$ 对称性，文献 [58] 指出它是反常的。不过利用莫雷-嘉当方程可使反常消失，因此基础理论中不存在反常，但复合 $U(1)$ 矢量场不能成为动力学场 [59]。

Superspace and Generalized Type IIB Supergravity

超空间与广义 IIB 型超引力

While the standard on-shell superspace constraints for Type IIB supergravity [50] are sufficient for the type IIB superstring action to be κ -symmetric [60], it was shown in [61] that they are not necessary. Instead, it was found that they yield a generalized version of type IIB supergravity equations of motion, bosonic part of which were found in [62], if the target space admits a Killing vector. The equations of motion involve Killing vector K and additional vector field Z with $K^\mu Z_\mu = 0$. For their description in exceptional field theory framework, see [63].

尽管 IIB 型超引力的标准在壳超空间约束 [50] 足以让 IIB 型超弦作用量满足 κ 对称性 [60]，但文献 [61] 已证明这些约束并非必要。研究发现，若目标空间容许一个基灵矢量，那么放宽约束后可得到 IIB 型超引力运动方程的广义形式，其玻色子部分已在文献 [62] 中给出。该运动方程包含基灵矢量 K 、额外矢量场 Z 以及 $K^\mu Z_\mu = 0$ 。关于它们在例外场论框架下的描述，参见文献 [63]。

Type I, I' and Heterotic Supergravities Coupled to Yang-Mills

I 型、I' 型与杂化超引力 (耦合杨-米尔斯场)

The fields of $N = (1,0)$ supergravity are $\left(e_\mu{}^a, B_{\mu\nu}, \phi, \psi_\mu, \chi\right)$, and those of the Yang-Mills multiplet are $\left(A_\mu, \lambda\right)$, where $\left(\psi_\mu, \lambda\right)$ are left-handed and χ is right-handed Majorana-Weyl. The coupling of the two multiplets was achieved long ago [64, 65] , and the bosonic part of Lagrangian in string frame, and setting $\kappa = 1$, is given by

$N = (1,0)$ 超引力的场为 $\left(e_\mu{}^a, B_{\mu\nu}, \phi, \psi_\mu, \chi\right)$ ，杨-米尔斯多重态的场为 $\left(A_\mu, \lambda\right)$ ，其中 $\left(\psi_\mu, \lambda\right)$ 是左手性，χ 是右手性马约拉纳-外尔费米子。两种多重态的耦合早已实现 [64, 65] ，弦框架下的拉格朗日玻色子部分在设定 $\kappa = 1$ 后可写为

$$\mathcal{L} = e e^{2\phi}\left[\frac{1}{4}R(\omega) + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}\beta F^I_{\mu\nu}F^{I\mu\nu}\right], \tag{15}$$

where $\beta = 1/g^2_{YM}$, and letting $F = F^I T^I$ with T^I in the adjoint representation,

其中 $\beta = 1/g^2_{YM}$ ，令 $F = F^I T^I$ ，且 T^I 属于伴随表示，

$$H = dB - \beta\,\mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \tag{16}$$

where $\mathrm{Tr}\left(T^I T^J\right) = -\delta^{IJ}$. The supertransformations of the fermions are

其中 $\mathrm{Tr}\left(T^I T^J\right) = -\delta^{IJ}$ 。费米子的超变换为

$$\delta\psi_\mu = D_\mu\left(\omega_+\right)\varepsilon,\ \delta\chi = \frac{1}{2}\gamma^\mu\varepsilon\partial_\mu\phi - \frac{1}{12}H_{\mu\upsilon\rho}\gamma^{\mu\upsilon\rho}\varepsilon, \tag{17}$$

where $\omega_{+\mu ab} = \omega_{\mu ab} + H_{\mu ab}$. In a celebrated paper [66] it was shown that the one loop gravitational, gauge, and mixed anomalies cancel only for gauge groups $SO\left(32\right)$, $E_8\times E_8$, $E_8\times U(1)^{248}$ and $U(1)^{496}$. Several years later, it was shown that the latter two theories do not have consistent coupling to quantum gravity, and as such they are in swampland [67, 68] . The case of $SO\left(32\right)$, together with the identification of β with the string tension α' , corresponds to the low energy effective action of Type I string. Performing the following field redefinitions in this action, with the primed field denoting the Type I fields (see, for example [69])

其中 $\omega_{+\mu ab} = \omega_{\mu ab} + H_{\mu ab}$ 。著名文献 [66] 已证明: 仅当规范群为 $SO\left(32\right)$, $E_8\times E_8$, $E_8\times U(1)^{248}$ 和 $U(1)^{496}$ 时，单圈引力反常、规范反常与混合反常会完全抵消。多年后研究表明，后两种理论无法与量子引力一致耦合，因此它们属于沼泽地 [67, 68] 。$SO\left(32\right)$ 的情形，在将 β 等同于弦张力 α' 后，对应 I 型弦的低能有效作用量。对该作用量做如下场重定义，带撇的场表示 I 型场 (参见例如文献 [69])

$$g'_{\mu\upsilon} = e^{-\phi}g_{\mu\upsilon},\ \phi' = -\phi,\ B'_{\mu\upsilon} = B_{\mu\upsilon},\ A'_\mu = A_\mu, \tag{18}$$

gives the low energy effective action of heterotic string with $SO\left(32\right)$ gauge symmetry [69]. These field redefinitions signify strong-weak coupling duality originally proposed in [70]. Taking the gauge group to be $E_8\times E_8$, it is related to $11D$ supergravity on a particular limit of $11D$ supergravity reduced on S^1/Z_2 . It has also been shown that the case of $E_8\times E_8$ in $10D$ is related to Type I$'$ (also known as Type IA) case, in which a suitable boundary term needs to be added to the effective action of the Type I case [71].

得到具有 $SO\left(32\right)$ 规范对称性的杂化弦低能有效作用量 [69]。这些场重定义对应最初由文献 [70] 提出的强弱耦合对偶。取规范群为 $E_8\times E_8$ 时，它关联到 $11D$ 超引力在 S^1/Z_2 约化极限下的形式。已有研究表明，$10D$ 中 $E_8\times E_8$ 的情形对应 I$'$ 型 (也称为 IA 型)，此时需要在 I 型有效作用量的基础上添加一个合适的边界项 [71]。

Dualization

对偶化

$N = (1, 0)$ supergravity plus Yang-Mills system in which a six-form potential, instead of the two-form potential, is employed was constructed directly long ago in [72]. The result may also be obtained by dualization. Note that both potentials describe the same number of degrees of freedom on-shell. Furthermore, the six-form potential couples to a five-brane, and as such it is reasonable to expect string-five-brane duality at work [73, 74] . Focusing on the supersymmetry aspects, the dualization of the two-form formulation proceeds by adding the following total derivative Lagrange multiplier term to Lagrangian (15)

$N=(1,0)$ 使用六形式势而非二形式势的超引力耦合杨-米尔斯系统早在文献 [72] 中就已直接构造完成，该结果也可通过对偶化得到。注意两种势在在壳条件下描述的自由度数量相同。此外，六形式势与五膜耦合，因此有理由认为弦-五膜对偶在其中发挥作用 [73, 74]。聚焦超对称性方面，对二形式表述进行对偶化的方法是在拉格朗日量 (15) 中添加如下全导数拉格朗日乘子项

$$\Delta\mathcal{L}=\frac{1}{2!\times 5!}\varepsilon^{\mu_1\cdots\mu_{10}}\left(\partial_{\mu_1}C_{\mu_2\ldots\mu_7}\right)\left(\partial_{\mu_8}B_{\mu_9\mu_{10}}\right)=\frac{1}{6}e\widetilde{G}^{\mu\nu\rho}\left(H_{\mu\nu\rho}+\beta X_{\mu\nu\rho}(A)\right), \tag{19}$$

where $C^{(6)}$ is the dual potential and $G^{(7)}=dC^{(6)}$. Treating H as an independent variable and integrating it out in the Lagrangian $\mathcal{L}_{\text{dual}}=\mathcal{L}+\Delta\mathcal{L}$ yields the theory in its dual formulation. Going to the fivebrane frame by letting $g_{\mu\nu}\to g'_{\mu\nu}=e^{-2\phi/3}g_{\mu\nu}$ yields

其中 $C^{(6)}$ 是对偶势，$G^{(7)}=dC^{(6)}$。将 H 视作独立变元并在拉格朗日量 $\mathcal{L}_{\text{dual}}=\mathcal{L}+\Delta\mathcal{L}$ 积分消去它，即可得到对偶表述形式的理论。通过令 $g_{\mu\nu}\to g'_{\mu\nu}=e^{-2\phi/3}g_{\mu\nu}$ 转换到五膜标架可得

$$\begin{aligned}\mathcal{L}_{\text{dual}}=&\,ee^{-2\phi/3}\left(\frac{1}{4}R-\frac{1}{2\times 7!}G_{\mu_1\ldots\mu_7}G^{\mu_1\cdots\mu_7}\right)-\frac{1}{4}\beta e\,\mathrm{Tr}\left(F_{\mu\upsilon}F^{\mu\upsilon}\right)\\&-\frac{4!\times 6!}{\beta}\varepsilon^{\mu_1\cdots\mu_{10}}C_{\mu_1\ldots\mu_6}\,\mathrm{Tr}\left(F_{\mu_7\mu_8}F_{\mu_9\mu_{10}}\right).\end{aligned} \tag{20}$$

In these conventions, the fermionic terms and supertransformations can be easily obtained from the expressions provided in [75]. Note the absence of $(\partial\phi)^2$ term in the Lagrangian. Diagonalizing the coupled field equations, the correct number of degrees of freedom follows nonetheless.

在此约定下，费米子项和超变换可很容易从文献 [75] 给出的表达式中得到。注意拉格朗日量中不存在 $(\partial\phi)^2$ 项，但即便如此，对角化耦合场方程后仍能得到正确数量的自由度。

Superspace

超空间

The superspace formulation of $N=(1,0)$ supergravity in $(10\mid 16)$ dimensional target superspace is useful in tackling variety of problems more conveniently then in component formulation, such as the worldsheet description of type I and heterotic string propagating in curved background. The pure $N=(1,0)$ supergravity in superspace was formulated long ago in [76], and with different set of conventional constraints in [77]. Coupling Yang-Mills fields as well, the superspace Bianchi identity for the super three-form field strength gets modified as $DH=\alpha'\,\mathrm{tr}\,F\wedge F$, and together with the super torsion constraints, it has been analyzed in [78], thereby providing a superspace formulation of heterotic supergravity in $10D$.

$N=(1,0)$ 超引力在 $(10\mid 16)$ 维目标超空间中的超空间表述，相较于分量表述能更方便地处理各类问题，例如 I 型和杂化弦在弯曲背景中传播的世界面描述。纯 $N=(1,0)$ 超引力的超空间表述早在文献 [76] 中就已给出，文献 [77] 中则给出了带有不同常规约束的形式。当同时耦合杨-米尔斯场时，超三形式场强的超空间 Bianchi 恒等式会修正为 $DH=\alpha'\,\mathrm{tr}\,F\wedge F$，结合超挠率约束，文献 [78] 对其进行了分析，最终给出了 $10D$ 中杂化超引力的超空间表述。

D = 9

Maximal 9D Gauged Supergravity

最大 9 维定标超引力

Maximal supergravity in $9D$ has the field content

$9D$ 维中的最大超引力的场内容为

$$\left\{e_\mu{}^a, \varphi, \tau, A_\mu^I, B_{\mu\upsilon}^i, C_{\mu\upsilon\rho}; \psi_\mu, \lambda, \tilde{\lambda}\right\}, \tag{21}$$

where $\tau \equiv \chi + ie^{-\varphi}$, and $I = (0, r)$, with $r, i = 1, 2$, the scalars (χ, φ) parametrize the coset $SL(2, \mathbb{R})/SO(2)$, the fermions are Dirac, and λ and $\tilde{\lambda}$ independent of each other. There is also a global scaling symmetry, $\mathbb{R}$, parametrized by the scalar ϕ , and the on-shell global trombone symmetry $\mathbb{R}^+$. The $SO(2)$, $SO(1,1)^+$ and $\mathbb{R}$ subgroups of $SL(2, \mathbb{R})$, $\mathbb{R}^+$, and a two dimensional non-abelian Lie group $A(1)$, were gauged, one at a time, in [79] by means of Scherk-Schwarz reductions of 11D, IIA, and IIB supergravities. These can be viewed as one-parameter deformations of the unique ungauged $N = 2$ supergravity. Considering a combination of these deformations, five two-parameter deformations were found in [79].

其中 $\tau \equiv \chi + ie^{-\varphi}$ ， $I = (0, r)$ ， 标量 (χ, φ) 在 $r, i = 1, 2$ 下参数化陪集 $SL(2, \mathbb{R})/SO(2)$ ，费米子为狄拉克费米子，且 λ 与 $\tilde{\lambda}$ 相互独立。此外还存在整体标度对称性 $\mathbb{R}$ ，由标量 ϕ 参数化，以及壳上整体长号对称性 $\mathbb{R}^+$ 。文献 [79] 通过施尔克-施瓦茨约化 11 维、IIA 和 IIB 超引力，分别对 $SL(2, \mathbb{R})$, $\mathbb{R}^+$ 的 $SO(2)$, $SO(1,1)^+$ 子群与 $\mathbb{R}$ 子群，以及一个二维非阿贝尔李群 $A(1)$ 进行了定标。这些都可以视为唯一无定标 $N = 2$ 超引力的单参数形变。通过考虑这些形变的组合，文献 [79] 还找到了五个双参数形变。

A direct analysis of gauging was later carried out in [80] where the embedding tensor formalism was employed, confirming these results. Denoting the generators of the algebra $SL(2, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}^+$ by T^A and using the notation $A_\mu^I = \left(A_\mu, A_\mu^r\right)$, one can introduce the embedding tensor $\theta_I{}^A$, so that the gauge group generator can be written as

后续文献 [80] 采用嵌入张量形式化对定标开展了直接分析，证实了上述结果。将代数 $SL(2, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}^+$ 的生成元记为 T^A ，并采用记号 $A_\mu^I = \left(A_\mu, A_\mu^r\right)$ ，我们可以引入嵌入张量 $\theta_I{}^A$ ，因此定标群生成元可以写为

$$X_I = \theta_I{}^A T_A, \; I = 1, 2, 3, \; A = 1, \ldots, 5. \tag{22}$$

Note that the magnetic fields (the duals of the electric ones listed in (21)) are not present in this gauging, though their representation content has been discussed in [80]. The closure of the supersymmetry algebra puts linear constraints on $\theta_I{}^A$ which are solved such that its independent components are [80]

请注意，此处定标不包含磁场 (即式 (21) 所列电场的对偶场)，不过其表示内容已在文献 [80] 中讨论。超对称代数的封闭性对 $\theta_I{}^A$ 施加了线性约束，求解后得到其独立分量为 [80]

$$\theta_0^m,\ \theta_0^5,\ \theta_r^4,\ \theta_r^5,\ m=1,2,3, r=1,2, \tag{23}$$

forming a triplet, two doublets and a singlets of $SL(2,\mathbb{R})$. Gauge invariance furthermore imposes the quadratic constraints [80]

构成 $SL(2,\mathbb{R})$ 的一个三重态、两个二重态和一个单态。此外，定标不变性还给出二次约束 [80]

$$\theta_0{}^m\left(12\theta_r{}^4+\theta_r{}^5\right)=0,\ \theta_r{}^4\theta_0{}^5=0,\ \theta_r{}^5\theta_0{}^5=0,$$

$$\theta_s{}^4\theta_0{}^m T_{mr}{}^s=0,\ \varepsilon^{rs}\theta_r{}^4\theta_s{}^5=0, \tag{24}$$

where $T_{mr}{}^s$ are 2×2 the representation matrices of the gauge generators T_A for $A=m$. The undeformed bosonic Lagrangian in these conventions, the covariant field strengths and the supertransformations, including the shift functions that arise in the supertransformations of the fermions, are given in [80]. See also [79], and both of these papers for the earlier references that cover specific gaugings of maximal $9D$ supergravity. The action [7] remains to be spelled out for the most general gauged maximal supergravity in $9D$. For a subset of these theories, the superpotential from which the potential can be deduced has been given in [79] where solutions including 1/2 BPS domain-wall, and maximally symmetric one with constant scalars, namely (A)dS and Minkowski spacetimes, are given.

其中 $T_{mr}{}^s$ 是定标生成元 T_A 对应 $A=m$ 的表示矩阵 2×2 。在该 conventions 下，未形变玻色拉格朗日量、协变场强以及包含费米子超变换中产生的移位函数在内的超变换，都已在文献 [80] 中给出。另见文献 [79]，这两篇论文都给出了涵盖最大 $9D$ 超引力特定定标的更早参考文献。对于 $9D$ 中最一般的定标最大超引力，作用量 [7] 仍有待明确给出。对于这类理论的一个子集，可推导出势能的超势已在文献 [79] 中给出，其中还给出了包含 1/2 BPS 畴壁的解，以及具有常标量的极大对称解，即 (反) 德西特时空与闵氏时空。

Half-Maximal 9D Supergravity Coupled to Vector Multiplets

耦合矢量多重态的半最大九维超引力

This theory was constructed by Noether procedure in [81], the result of which we summarize below. Combining n -copies of the Maxwell multiplet consisting of the fields $\left(A_\mu,\phi,\lambda\right)$ with the supergavity multiplet gives the field content

该理论通过诺特定理程序在文献 [81] 中构造完成，我们在下方总结其结果。将包含场量 $\left(A_\mu,\phi,\lambda\right)$ 的 n 份麦克斯韦多重态与超引力多重态结合，即可得到场内容

$$\left\{e_\mu{}^a, B_{\mu\nu}, \phi, \mathcal{V}_M{}^A, A_\mu^M; \psi_\mu, \chi, \lambda^a\right\}, \tag{25}$$

where $M,A=0,1,\dots,n, a=1,\dots,n, \varphi$ is the dilaton, $\mathcal{V}_M^A=\left(\mathcal{V}_M{}^0,\mathcal{V}_M{}^a\right)$ is the $SO(n,1)/SO(n)$ coset representative, and the fermions are pseudo-Majorana. The vector A_μ^0 belongs to supergravity multiplet. The $SO(n,1)$ invariant tensor $\eta=\mathrm{diag}(-1,+1,\dots,+1)$ and the positive definite scalar matrix $\mathcal{M}$ are given by

其中 $M, A = 0, 1, \ldots, n, a = 1, \ldots, n, \varphi$ 是 dilaton(伸缩子)，$\mathcal{V}_M^A = \left(\mathcal{V}_M{}^0, \mathcal{V}_M{}^a\right)$ 是 $SO(n,1)/SO(n)$ 陪集代表元，费米子为伪马约拉纳费米子。矢量场 A_μ^0 属于超引力多重态。$SO(n,1)$ 不变张量 $\eta = \mathrm{diag}(-1, +1, \ldots, +1)$ 与正定标量矩阵 $\mathcal{M}$ 由下式给出

$$\eta_{MN} = -\mathcal{V}_M{}^0\mathcal{V}_N{}^0 + \mathcal{V}_M{}^a\mathcal{V}_N{}^a,\ \mathcal{M}_{MN} = \mathcal{V}_M{}^0\mathcal{V}_N{}^0 + \mathcal{V}_M{}^a\mathcal{V}_N{}^a, \tag{26}$$

and the field strengths are defined as

场强定义如下

$$P_\mu^a = \mathcal{V}_0^I\partial_\mu\mathcal{V}_I^a,\ F_{\mu\upsilon}^M = 2\partial_{[\mu}A_{\upsilon]}^M,\ H_{\mu\upsilon\rho} = 3\partial_{[\mu}B_{\upsilon\rho]} + 3\eta_{MN}F_{[\mu\upsilon}^M A_{\rho]}^N.$$

(27)

The bosonic part of the ungauged Lagrangian is given by [81]

未加规范作用拉氏量的玻色子部分由文献 [81] 给出

$$e^{-1}\mathcal{L} = -\frac{1}{4}R + \frac{7}{4}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{12}e^{-4\varphi}H_{\mu\upsilon\rho}H^{\mu\upsilon\rho} - \frac{1}{4}e^{-2\varphi}\mathcal{M}_{MN}F_{\mu\upsilon}^M F^{\mu\upsilon N}$$
$$+\frac{1}{2}P_\mu^a P_a^\mu \tag{28}$$

and the supertransformations of the fermions are

且费米子的超变换为

$$\delta\psi_\mu = D_\mu\varepsilon - \frac{1}{14\sqrt{2}}e^{-\phi}\mathcal{V}_M{}^0F_{\rho\sigma}^M\left(\gamma_\mu{}^{\rho\sigma} - 12\delta_\mu^\rho\gamma^\sigma\right)\varepsilon$$
$$+\frac{1}{42}e^{-2\phi}H_{\rho\sigma\tau}\left(\gamma_\mu{}^{\rho\sigma\tau} - \frac{15}{2}\delta_\mu^\rho\gamma^{\sigma\tau}\right)\varepsilon$$

$$\delta\chi = -\frac{\sqrt{7}}{2}\gamma^\mu\varepsilon\partial_\mu\phi + \frac{1}{6\sqrt{7}}e^{-2\phi}H_{\mu\upsilon\rho}\gamma^{\mu\upsilon\rho}\varepsilon - \frac{1}{2\sqrt{14}}e^{-\phi}\mathcal{V}_M{}^0F_{\mu\upsilon}^M\gamma^{\mu\upsilon}\varepsilon,$$

$$\delta\lambda^a = -\frac{1}{\sqrt{2}}\gamma^\mu P_\mu^a\varepsilon - \frac{1}{2\sqrt{2}}e^{-\phi}\mathcal{V}_M{}^aF_{\mu\upsilon}^M\gamma^{\mu\upsilon}\varepsilon. \tag{29}$$

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[7] In the case of gauged trombone symmetry, the theory can be formulated only at the level of equations of motion, since the undeformed Lagrangian scales under this global symmetry. For a detailed treatment of gauging the trombone symmetries, see [8].

[7] 对于规范变换长号对称性情形，该理论只能在运动方程层面表述，因为未形变拉格朗日量在该整体对称性下会发生标度变换。关于长号对称性规范变换的详细处理，参见文献 [8]。

The theory also has the (off-shell) global $SO(1,1)$ scaling symmetry under which $\phi \to \phi+\alpha, A_\mu^M \to e^\alpha A_\mu^M$ and $B_{\mu\nu} \to e^{2\alpha} B_{\mu\nu}$, not to be confused with the (on-shell) trombone symmetry mentioned in Footnote 2. The gauging of a subgroup of $SO(1,1)\times SO(n,1)$ by employing the $(n+1)$ gauge field present in the theory has not been carried out so far, but it is expected to be very similar to the gauging of a subgroup of $SO(1,1)\times SO(n,5)$ in $N=4, 5D$ supergravity [82] summarized in section " $N=4$ Supergravity Coupled to Vector Multiplets in $5D$ ".

该理论还具有 (离壳) 整体 $SO(1,1)$ 标度对称性，在此对称性下 $\phi \to \phi+\alpha, A_\mu^M \to e^\alpha A_\mu^M$ 和 $B_{\mu\nu} \to e^{2\alpha} B_{\mu\nu}$ 成立，请勿与脚注 2 中提及的 (在壳) 长号对称性混淆。目前尚未通过利用该理论中现存的 $(n+1)$ 规范场完成对 $SO(1,1)\times SO(n,1)$ 子群的规范耦合，但预期该过程与" $N=4$ 维超引力耦合矢量多重态在 $5D$ " 小节中总结的 $N=4, 5D$ 超引力里 $SO(1,1)\times SO(n,5)$ 子群的规范耦合非常相似 [82]。

$D=8$

Maximal 8D Gauged Supergravity

极大 8 维规范超引力

This theory was constructed in [83] by Scherk-Schwarz reduction of $11D$ super-gravity on $SU(2)$ group manifold, and it gives a gauged maximal supergravity in $9D$ with scalars parametrizing the coset $SL(3,\mathbb{R})/SO(3)\times SL(2,\mathbb{R})/SO(2)$, and local gauge symmetry $SU(2)\rtimes\mathbb{R}^3$. The Lagrangian, the supertransformations, the definition of field strengths, and the gauge transformations are given in [83]. In this result the gauge group $SO(3)$ is the maximal compact subgroup of $SL(3,\mathbb{R})$ which employs three of the six vector fields present in the maximal supergravity multiplet. The six gauge fields can be assembled into $(3,2)$ representation of $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$, and the general gaugings can thus be studied systematically by means of the embedding tensor formalism, and this was done in [84, 85], as we summarize below. The supergravity multiplet is supplemented by dual three-form S^α and four-form W^M required by the tensor hierarchy associated with the embedding tensor formalism, without upsetting the total count of physical degrees of freedom such that the total set of fields are

该理论由文献 [83] 通过将 $11D$ 超引力在 $SU(2)$ 群流形上进行 Scherk-Schwarz 约化构造得到，它给出了 $9D$ 维的极大规范超引力，其中标量参数化陪集 $SL(3,\mathbb{R})/SO(3)\times SL(2,\mathbb{R})/SO(2)$ ，局部规范对称性为 $SU(2)\rtimes\mathbb{R}^3$ 。拉格朗日量、超变换、场强定义以及规范变换均在文献 [83] 中给出。在此结果中，规范群 $SO(3)$ 是 $SL(3,\mathbb{R})$ 的极大紧子群，它利用了极大超引力多重态中存在的六个矢量场中的三个。六个规范场可组装为 $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$ 的 $(3,2)$ 表示，因此一般规范作用可以通过嵌入张量形式体系进行系统研究，相关工作已在文献 [84, 85] 中完成，我们将在下文总结。超引力多重态由嵌入张量形式体系关联的张量层次结构要求引入对偶三形式场 S^α 和四形式场 W^M ，且不会改变物理自由度的总数量，因此完整的场集合为

$$\{e_\mu{}^a, \phi, B, \mathcal{V}_M{}^i, A_\mu^{\alpha M}, B_{\mu\nu M}, C_{\mu\nu\rho}{}^\alpha, S_{\mu\nu\rho}{}^\alpha, W_{\mu\nu\rho\sigma}{}^M; \psi_\mu, \chi_i^A\}, \tag{30}$$

where the 3×3 matrix $\mathcal{V}_M{}^i$ is the representative of the $SL(3,R)/SO(3)$ coset, the scalars (B,ϕ) parametrize the coset $SL(2,\mathbb{R})/SO(2)$, the index $A=1,2$, and the fermions are pseudo-Majorana. The index $\alpha=1,2$

labels the $SL(2,\mathbb{R})$ doublet. The fields S^α and W^M are dually related to the ordinary supergravity multiplet fields C_α and B_M , respectively. Duality equations are to be imposed by hand, and they ensure that the physical degrees of freedom remain as $128_B + 128_F$. The most general gauging is governed by the embedding tensors $\theta_{\alpha M,K}{}^L$ and $\theta_{\alpha M,\beta}{}^\gamma$. Thus the gauge group generators $G \subset SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ can be written as

其中 3×3 矩阵 $\mathcal{V}_M{}^i$ 是 $SL(3,R)/SO(3)$ 陪集的代表元，标量 (B,ϕ) 参数化陪集 $SL(2,\mathbb{R})/SO(2)$ ，指标满足 $A = 1,2$ ，费米子为伪马约拉纳费米子。指标 $\alpha = 1,2$ 标记 $SL(2,\mathbb{R})$ 二重态。场 S^α 和 W^M 分别对偶于普通超引力多重态场 C_α 和 B_M 。对偶方程需要手动引入，它们保证物理自由度保持为 $128_B + 128_F$ 。最一般的规范作用由嵌入张量 $\theta_{\alpha M,K}{}^L$ 和 $\theta_{\alpha M,\beta}{}^\gamma$ 描述。因此规范群生成元 $G \subset SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ 可写为

$$X_{\alpha M} = \theta_{\alpha M,K}{}^L t_L{}^K + \theta_{\alpha M,\beta}{}^\gamma t_\gamma{}^\beta. \tag{31}$$

As a consequence of the linear and quadratic constraints on the embedding tensors, it has been shown that they can be parametrized by $\xi_{\alpha M}$ and $f_\alpha{}^{MN} = f_\alpha{}^{(MN)}$ as follows [84]

由于嵌入张量满足线性约束和二次约束，已有结果表明它们可通过 $\xi_{\alpha M}$ 和 $f_\alpha{}^{MN} = f_\alpha{}^{(MN)}$ 参数化如下 [84]

$$\theta_{\alpha M},{}^\beta{}_\gamma = \xi_{\gamma M}\delta^\beta_\alpha - \frac{1}{2}\delta^\beta_\gamma \xi_{\alpha M},$$

$$\theta_{\alpha M},{}^K{}_L = f_\alpha{}^{KR}\varepsilon_{RML} - \frac{3}{4}\left(\xi_{\alpha L}\delta^K_M - \frac{1}{3}\xi_{\alpha M}\delta^K_L\right), \tag{32}$$

with the conditions that

满足条件

$$\xi_{\alpha[M}\xi_{\beta|N]} = 0,\ f^{MN}_\alpha \xi_{\beta N} = 0,\ \varepsilon^{\alpha\beta}\left(f^{MK}_\alpha f_\beta{}^{NL}\varepsilon_{PKL} - f_\alpha{}^{MN}\xi_{\beta P}\right) = 0.$$

(33)

For a discussion of the solutions to the constrains which determine the possible gauge groups, see [84]. They include the 1-dimensional subgroups of $SL(2,\mathbb{R})$ (rescalings, shifts, and the Borel subgroup), and the Borel subgroup of $SL(3,\mathbb{R})$. The field strengths for the fields (A,B,C) , see (30), are given in [84], but not the action and supersymmetry transformations. In [86], where the embedding tensor is also employed, the focus is on the $SO(3)$ gauging, and it is shown that this gauging, which makes use of A^{1M}_μ obtained by the Scherk-Schwarz reduction of $11D$ supergravity [83], is related by an $SL(2,\mathbb{R})$ transformation to the gauging obtained by A^{2M}_μ , obtained from the dimensional reduction of massive 11D supergravity [87].

有关确定可能规范群的约束解的讨论，参见文献 [84]。这些约束解包括 $SL(2,\mathbb{R})$ 的一维子群 (重标度变换、平移和博雷尔子群) 以及 $SL(3,\mathbb{R})$ 的博雷尔子群。场 (A,B,C) 的场强 (参见式 (30)) 已在文献 [84] 中给出，但未给出作用量和超对称变换。文献 [86] 同样采用了嵌入张量，该文献聚焦于 $SO(3)$ 规范，文中表明，该规范利用了通过对 $11D$ 超引力做 Scherk-Schwarz 约化得到的 A^{1M}_μ [83]，它通过一个 $SL(2,\mathbb{R})$ 变换与 A^{2M}_μ 得到的规范相关联，而 A^{2M}_μ 本身是从有质量 11 维超引力维约化得到的 [87]。

Half-Maximal 8D Gauged Supergravity Coupled to Vector Multiplets

耦合矢量多重态的半极大 8 维定规超引力

This theory was constructed by Noether procedure in [88] where the $(n+2)$ parameter subgroup of the global $SO(n,2)$ symmetry was gauged. We summarize these results next. Combining n -copies of the vector multiplet consisting of the fields $\left(A_\mu, 2\phi, \lambda\right)$ with the supergavity multiplet gives the field content

该理论通过诺特 procedure 于文献 [88] 中构造完成，其中整体对称性 $SO(n,2)$ 的 $(n+2)$ 参数子群被定域规范化。我们在下文中总结这些结果。将 n 份由场量 $\left(A_\mu, 2\phi, \lambda\right)$ 构成的矢量多重态与超引力多重态结合，即可得到场内容

$$\left\{e_\mu{}^r, B_{\mu\nu}, \varphi, \mathcal{V}_M{}^A, A_\mu^M; \psi_\mu, \chi, \lambda^a\right\}, \tag{34}$$

where $M, A = 1, \ldots, n+2, a = 1, \ldots, n$, the scalar φ is the dilaton, $\mathcal{V}_M{}^A$ is the representative of the $SO(n,2)/SO(n) \times SO(2)$ coset, and the fermions are pseudo-Majorana. Writing $A_\mu^M = \left(A_\mu^m, A_\mu^\alpha\right)$ with $m = 1,2$ and $\alpha = 1, \ldots, n$, the vectors A_μ^m belong to the supergravity multiplet. The $SO(n,2)$ invariant tensor $\eta = \text{diag}(-1,-1,+1,\ldots,+1)$ and the positive definite scalar matrix $\mathcal{M}$ are given by

其中 $M, A = 1, \ldots, n+2, a = 1, \ldots, n$ ，标量场 φ 为伸缩子，$\mathcal{V}_M{}^A$ 是 $SO(n,2)/SO(n) \times SO(2)$ 陪集的代表元，费米子为伪马约拉纳费米子。在写出含 $m = 1,2$ 和 $\alpha = 1, \ldots, n$ 的 $A_\mu^M = \left(A_\mu^m, A_\mu^\alpha\right)$ 后，矢量场 A_μ^m 属于超引力多重态。$SO(n,2)$ 不变张量 $\eta = \text{diag}(-1,-1,+1,\ldots,+1)$ 与正定标量矩阵 $\mathcal{M}$ 由下式给出

$$\eta_{MN} = -\mathcal{V}_M{}^i \mathcal{V}_N^i + \mathcal{V}_M{}^a \mathcal{V}_N{}^a, \ \mathcal{M}_{MN} = \mathcal{V}_M{}^i \mathcal{V}_N^i + \mathcal{V}_M{}^a \mathcal{V}_N{}^a, \tag{35}$$

where $i = 1,2$. Using all the vector fields to gauge a $(n+2)$ parameter semisimple subgroup $G \subset SO(n,2)$, we introduce the field strengths

其中 $i = 1,2$ 。利用所有矢量场规范一个 $(n+2)$ 参数半单子群 $G \subset SO(n,2)$ ，我们引入场强

$$\mathcal{F}_{\mu\nu}^I = 2\partial_{[\mu} A_{\nu]}^I + f_{KL}{}^I A_\mu^K A_\nu^L$$

$$\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} + 3\mathcal{F}_{[\mu\nu}^I A_{\rho]}^J \eta_{IJ} - f_{IJ}{}^L \eta_{LK} A_{[\mu}^I A_\nu^J A_{\rho]}^K, \tag{36}$$

with $f_{IJ}{}^K$ representing structure constants of the gauge group G . The gauged scalar current is given by

其中 $f_{IJ}{}^K$ 代表规范群 G 的结构常数。定标标量流由下式给出

$$\mathcal{V}^M{}_A \left(\partial_\mu \delta_M^N + f_{MP}{}^N A_\mu^P\right) \mathcal{V}_N{}^B = \left(\mathcal{Q}_{\mu a}{}^b \mathcal{P}_a{}^i\right). \tag{37}$$

Using the above ingredients, the Lagrangian is given by [8] [88]

利用上述要素，拉格朗日量可写为 [8] [88]

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{1}{4}e^{\phi}\mathcal{M}_{MN}\mathcal{F}^{M}_{\mu\nu}\mathcal{F}^{\mu\nu N} - \frac{1}{12}e^{2\phi}\mathcal{H}_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{3}{8}\partial_\mu\phi\partial^\mu\phi$$

$$+\frac{1}{4}\mathcal{P}^{ai}_{\mu}\mathcal{P}^{\mu}_{ai} - \frac{1}{8}e^{-\phi}C^a C_a \tag{38}$$

where

其中

$$C^a = f_{IJ}{}^{K}V^{I}{}_{1}V^{J}{}_{2}V_{K}{}^{a}, \tag{39}$$

and the supertransformations of the fermions are

且费米子的超变换为

$$\delta\psi_\mu = D_\mu\varepsilon + \frac{i}{12\sqrt{2}}\mathcal{F}^{I}_{\rho\sigma}\left(\mathcal{V}_M{}^1 + i\gamma_9\mathcal{V}_M{}^2\right)\left(\gamma_\mu{}^{\rho\sigma} - 10\delta^{\rho}_{\mu}\gamma^{\sigma}\right)\varepsilon \tag{40}$$

$$-\frac{1}{36}e^{\phi}\mathcal{H}_{\rho\sigma\tau}\left(\gamma_\mu{}^{\rho\sigma\tau} - 6\delta^{\rho}_{\mu}\gamma^{\sigma\tau}\right)\varepsilon,$$

$$\delta\chi = -\frac{1}{2}i\partial_\mu\phi\gamma^\mu\varepsilon + \frac{1}{6\sqrt{2}}e^{\phi/2}\mathcal{F}^{I}_{\rho\sigma}\left(\mathcal{V}_M{}^1 + i\gamma_9\mathcal{V}_M{}^2\right)\gamma^{\mu\upsilon}\varepsilon + \frac{1}{18}ie^{\phi}\mathcal{H}_{\mu\upsilon\rho}\gamma^{\mu\upsilon\rho}\varepsilon,$$

$$\delta\lambda^a = -\frac{i}{2}\gamma^\mu\left(\mathcal{P}^{a1}_{\mu} + i\gamma^9\mathcal{P}^{a2}_{\mu}\right)\varepsilon + \frac{1}{2\sqrt{2}}e^{\phi/2}\mathcal{F}^{I}_{\mu\upsilon}L_I{}^a\gamma^{\mu\upsilon}\varepsilon + \frac{i}{2\sqrt{2}}e^{-\phi/2}C^a\gamma_9\varepsilon,$$

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[8] The ungauged theory also has the (off-shell) global $SO(1,1)$ scaling symmetry under which $\phi \to \phi - \alpha, A^M_\mu \to e^{\alpha}A^M_\mu$ and $B_{\mu\nu} \to e^{2\alpha}B_{\mu\nu}$.

[8] 未规范理论还具有 (离壳) 整体 $SO(1,1)$ 标度对称性，在此对称性下 $\phi \to \phi - \alpha, A^M_\mu \to e^{\alpha}A^M_\mu$ 和 $B_{\mu\nu} \to e^{2\alpha}B_{\mu\nu}$ 。

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where $D_\mu\varepsilon^i = D_\mu(\omega, Q)\varepsilon^i$. The terms arising due to the gauging are remarkably similar to those we shall see in $6D, N = (1,0)$ supergravity. In particular a positive potential, which plays a key role in Minkowski compactification, arises in both cases.

其中 $D_\mu\varepsilon^i = D_\mu(\omega, Q)\varepsilon^i$ 。由定规范产生的项与我们将在 $6D, N = (1,0)$ 超引力中见到的项极为相似。尤其是，在两种情形中都会产生对闵氏紧致化起到关键作用的正势能。

The most general gaugings of the half-maximal supergravity coupled to arbitrary number of vector multiplets remains to be worked out. The embedding tensor formalism may be employed directly in $7D$ to this end. In a search for higher dimensional origin, on the other hand, it is natural to consider the framework of the so-called double field theory; see section ”Supergravities in Extended Geometry Framework”. For example, the Lagrangian (38) agrees with the result in [89, eq. (3.43)] obtained in a particular reduction of the (gauged) double field theory.

耦合任意数量向量多重态的半极大超引力的最一般规范作用仍有待推导。为此，可以直接在 $7D$ 中采用嵌入张量形式化方法。另一方面，在探寻其更高维起源时，很自然会考虑所谓的双场理论框架；参见“扩展几何框架下的超引力”一节。例如，拉格朗日量 (38) 与 (规范) 双场理论特定约化下得到的结果 [89, 式 (3.43)] 一致。

D = 7

Maximal 7D Gauged Supergravity

最大 7 维定标超引力

The ungauged maximal supergravity in $7D$ was constructed by Noether procedure in [90], for the supergravity multiplet

$7D$ 中的非定标最大超引力通过诺特方法构造于文献 [90]，针对超引力多重态

$$\left(e_\mu{}^m, \mathcal{V}_M{}^{ab}, A_\mu^{MN}, B_{\mu\nu M}; \psi_\mu^a, \chi^{abc}\right), \tag{41}$$

where $\mathcal{V}_M{}^{ab} = \mathcal{V}_M{}^{[ab]}$ is representative of coset $SL(5)/SO(5)$, with $M = 1, \dots, 5$ and $a = 1, \dots, 4$ labeling 4-plet of $USp(4) \approx SO(5)$. The gauge fields A_μ^{MN} are in the 10-plet of $SL(5)$. The spinors are symplectic Majorana, and $\chi^{abc} = \chi^{[ab]c}, \chi^{[abc]} = 0, \Omega_{ab}\chi^{abc} = 0$ is in the 16-plet of $USp(4)$.

其中 $\mathcal{V}_M{}^{ab} = \mathcal{V}_M{}^{[ab]}$ 是陪集 $SL(5)/SO(5)$ 的代表元，$M = 1, \dots, 5$ 和 $a = 1, \dots, 4$ 标记 $USp(4) \approx SO(5)$ 的 4 重态。规范场 A_μ^{MN} 属于 $SL(5)$ 的 10 重态。旋量是辛马约拉纳旋量，且 $\chi^{abc} = \chi^{[ab]c}, \chi^{[abc]} = 0, \Omega_{ab}\chi^{abc} = 0$ 属于 $USp(4)$ 的 16 重态。

It was noted in [90] that the 10 gauge fields could not be used to gauge $SO(5) \subset SL(5)$ because the two-form potentials carry a non-trivial representation of $SL(5)$. Later, using instead the multiplet

文献 [90] 中指出，无法用这 10 个规范场对 $SO(5) \subset SL(5)$ 定标，因为二形式势携带 $SL(5)$ 的非平凡表示。随后，研究者改用满足形式为 $SO(5) \subset SL(5)$ 的“自对偶”条件的三形式势的多重态，

$$\left(e_\mu{}^m, \mathcal{V}_M{}^{ab}, A_\mu^{MN}, C_{\mu\nu\rho}{}^M; \psi_\mu^a, \chi^{abc}\right), \tag{42}$$

with the three-form potential satisfying a ”self-duality” condition of the form $dC^M = m \star C^M$, where m is the gauge coupling constant, the $SO(5)$ gauging was achieved by Noether procedure [91]. This construction was motivated by the compactification of $11D$ supergravity on the sphere S^4 [92]. Subsequently, $SO(4,1)$ and $SO(3,2)$ gaugings were obtained [93]. A unified treatment which yields more general gaugings was achieved in [94], using the embedding tensor formalism for the following multiplet of fields

三形式势满足形式为 $dC^M = m \star C^M$ 的“自对偶”条件，其中 m 为规范耦合常数，该 $SO(5)$ 规范化通过诺特定理步骤实现 [91]。该构造的动机是 $11D$ 超引力在球面 S^4 上的紧化 [92]。随后，研究者得到了 $SO(4,1)$ 和 $SO(3,2)$ 规范化 [93]。文献 [94] 借助嵌入张量形式，对如下场多重态给出了能得到更一般规范化的统一处理

$$\left(e_\mu{}^m, \mathcal{V}_M{}^{ab}, A_\mu^{MN}, B_{\mu\upsilon M}, C_{\mu\upsilon\rho}{}^M; \psi_\mu^a, \chi^{abc}\right). \tag{43}$$

Both the two-form $B_{\mu\upsilon M}$ and the 3-form $C_{\mu\upsilon\rho}{}^M$ are now present but, as we shall see below, the C -field equation will relate their field strengths to each other, and consequently the on-shell degrees of freedom remain as $128_B + 128_F$. In the rest of this subsection, we follow [94] to summarize their key results. The most general gauging is encoded in a real embedding tensor $\theta_{MN,P}{}^Q = \theta_{[MN],P}{}^Q$, which determines the gauge group $G \subset SL(5)$ with generators

现在二形式 $B_{\mu\upsilon M}$ 和三形式 $C_{\mu\upsilon\rho}{}^M$ 都存在，但正如我们下文将看到的，C 场方程会将二者的场强相互关联，因此壳自由度数目仍和 $128_B + 128_F$ 一致。在本小节余下部分，我们遵循文献 [94] 总结其核心结果。最一般的定标由实嵌入张量 $\theta_{MN,P}{}^Q = \theta_{[MN],P}{}^Q$ 编码，它确定了带有如下生成元的规范群 $G \subset SL(5)$:

$$X_{MN} = \theta_{MN,P}{}^Q t^P{}_Q, \tag{44}$$

where $t^P{}_O$ are the $SL(5)$ generators. Thus, the covariant derivatives are given by $D_\mu = \nabla_\mu - gA_\mu^{MN} X_{MN}$. Supersymmetry requirement imposes a linear constraint on the embedding tensor such that in the product $\mathbf{10} \otimes \mathbf{24}$, only the representations $\mathbf{15} + \overline{\mathbf{40}}$ survive. Therefore, it can be parametrized as

其中 $t^P{}_O$ 是 $SL(5)$ 生成元。因此，协变导数由 $D_\mu = \nabla_\mu - gA_\mu^{MN} X_{MN}$ 给出。超对称要求对嵌入张量施加一个线性约束，使得在乘积 $\mathbf{10} \otimes \mathbf{24}$ 中，只有表示 $\mathbf{15} + \overline{\mathbf{40}}$ 保留下来。因此，嵌入张量可以参数化为:

$$\theta_{MN,P}{}^Q = \delta^Q_{[M} Y_{N]P} - 2\varepsilon_{MNPRS} Z^{RS,Q}, \tag{45}$$

where $Y_{MN} = Y_{(MN)}$, and $Z^{MN,P} = Z^{[MN],P}$ with $Z^{[MN,P]} = 0$. In addition, a quadratic constraint needs to be imposed on the embedding tensor to ensure the closure of the gauge algebra. It has been shown that the full content of this quadratic constraint is encoded in the equation [94]

其中 $Y_{MN} = Y_{(MN)}$ ，且 $Z^{MN,P} = Z^{[MN],P}$ 满足 $Z^{[MN,P]} = 0$ 。此外，需要对嵌入张量施加二次约束以保证规范代数封闭。已有证明该二次约束的全部内容都包含在下述方程中 [94]

$$Z^{MN,P} X_{MN} = 0. \tag{46}$$

The building blocks for the action are the covariantly transforming field strengths

该作用量的构造模块是协变变换的场强

$$\mathcal{F}_{\mu\upsilon}{}^{MN} = 2\partial_{[\mu} A_{\upsilon]}^{MN} + g\left(X_{PQ}\right)_{RS}{}^{MN} A_{[\mu}^{PQ} A_{\upsilon]}^{RS} + gZ^{MN,P} B_{\mu\upsilon P},$$

$$\mathcal{H}_{\mu\upsilon\rho M} = 3D_{[\mu}B_{\upsilon\rho]M} + 6\varepsilon_{MNPQR}A^{NP}_{[\mu}\left(\partial_\upsilon A^{QR}_{\rho]} + \frac{2}{3}gX_{ST,U}{}^{Q}A^{RU}_{\upsilon}A^{ST}_{\rho]}\right)$$

$$+gY_{MN}S^{N}_{\mu\upsilon\rho}$$

$$Y_{MN}\mathcal{G}_{\mu\upsilon\rho\sigma}{}^{N} = Y_{MN}\left(4D_{[\mu}C_{\upsilon\rho\sigma]}{}^{N} + 6\mathcal{F}^{NP}_{[\mu\upsilon}B_{\rho\sigma]P}\right.$$

$$+3gZ^{NP,Q}B_{[\mu\upsilon P}B_{\rho\sigma]Q} + 8\varepsilon_{PQRST}A^{NP}_{[\mu}A^{QR}_{\upsilon}\partial_\rho A^{ST}_{\sigma]}$$

$$\left.+4g\varepsilon_{PQRVW}X_{ST,U}{}^{V}A^{NP}_{[\mu}A^{QR}_{\upsilon}A^{ST}_{\rho}A^{UW}_{\sigma]}\right), \tag{47}$$

where $\left(X_{PQ}\right)_{RS}{}^{MN} = 2\left(X_{PQ}\right)_{[R}{}^{[M}\delta^{N]}_{S]}$ and $(X_{MN})_P{}^Q = \theta_{MN,P}{}^Q$. Furthermore, the 4-form field strength arises in the action and transformation rules always under the projection with Y_{MN} . Denoting the gauge transformations associated with the (A,B,C) fields by $(\Lambda,\sum,\Phi)$, the gauge algebra takes the form $\left[\delta_{\Lambda_1},\delta_{\Lambda_2}\right] = \delta_\Lambda + \delta_\Xi + \delta_\Phi$ and $\left[\delta_{\Xi_1},\delta_{\Xi_2}\right] = \delta_\Phi$, as shown in [94].

其中 $\left(X_{PQ}\right)_{RS}{}^{MN} = 2\left(X_{PQ}\right)_{[R}{}^{[M}\delta^{N]}_{S]}$ 和 $(X_{MN})_P{}^Q = \theta_{MN,P}{}^Q$ 。此外，4 形式场强总会在作用量和变换规则中伴随 Y_{MN} 投影出现。将对应于 (A,B,C) 场的规范变换记为 $(\Lambda,\sum,\Phi)$ ，则规范代数形如 $\left[\delta_{\Lambda_1},\delta_{\Lambda_2}\right] = \delta_\Lambda + \delta_\Xi + \delta_\Phi$ 和 $\left[\delta_{\Xi_1},\delta_{\Xi_2}\right] = \delta_\Phi$ ，如文献 [94] 所示。

Few remaining building blocks are constructed as follows. Firstly, the SL(5) Lie algebra valued gauged Maurer-Cartan form decomposes as

剩余少数构造模块构造如下: 首先，SL(5) 李代数取值的规范化 Maurer-Cartan 形式可分解为

$$\mathcal{V}_{ab}{}^{M}\left(\partial_\mu\mathcal{V}_M{}^{cd} - gA_\mu{}^{PQ}X_{PQ,M}{}^{N}\mathcal{V}_N{}^{cd}\right) = P_{\mu ab}{}^{cd} + 2Q_{\mu[a}{}^{[c}\delta^{d]}_{b]}. \tag{48}$$

As usual, $P_{\mu ab}{}^{cd}$ facilitates construction of a kinetic term for the scalars, and $Q_{\mu a}{}^{b}$ is the composite $SO(5)$ connection which is needed in the covariant derivatives of the fermions.

和通常情况一样，$P_{\mu ab}{}^{cd}$ 可用来构造标量的动能项，而 $Q_{\mu a}{}^{b}$ 是复合 $SO(5)$ 联络，是费米子协变导数所必需的。

Another building block for gauging is the T -tensor, which is the embedding tensor suitably dressed up with coset vielbeins. More specifically, using the decomposition of its $SL(5)$ content under $USp(4)$ as $\mathbf{15}+\overline{\mathbf{40}} \rightarrow (\mathbf{1}+\mathbf{14})+(\mathbf{5}+\mathbf{35})$, one defines the corresponding T -tensors denoted by $B, B^{[ab]}{}_{[cd]}, C_{[ab]}$ and $C^{[ab]}{}_{(cd)}$ as

规范化的另一构造模块是 T 张量，它是用陪集标架装配后的嵌入张量。更具体地说，将其 $SL(5)$ 分量按 $USp(4)$ 分解为 $\mathbf{15}+\overline{\mathbf{40}} \rightarrow (\mathbf{1}+\mathbf{14})+(\mathbf{5}+\mathbf{35})$ ，即可定义相应的 T 张量，记为 $B, B^{[ab]}{}_{[cd]}, C_{[ab]}$ 和 $C^{[ab]}{}_{(cd)}$ ，即

$$Y_{MN} = \mathcal{V}_M{}^{ab}\mathcal{V}_N{}^{cd}Y_{ab,cd},\ Z^{MN,P} = \sqrt{2}\mathcal{V}_{ab}{}^{M}\mathcal{V}_{cd}{}^{N}\mathcal{V}_{ef}{}^{P}\Omega^{bd}Z^{(ac)[ef]},$$

(49)

where the T-tensor fields are decomposed as

其中 T 张量场被分解为

$$Y_{ab,cd} = \frac{1}{\sqrt{2}}\Big[\Big(\Omega_{ac}\Omega_{bd} - \frac{1}{4}\Omega_{ab}\Omega_{cd}\Big)B + \Omega_{ae}\Omega_{bf}B^{[ef]}{}_{[cd]},$$

$$Z^{(ab)[cd]} = \frac{1}{16}\Omega^{a[c}C^{d]b} + \frac{1}{16}\Omega^{b[c}C^{d]a} - \frac{1}{8}\Omega^{ae}\Omega^{bf}C^{cd}{}_{ef}. \tag{50}$$

The bosonic part of gauged supergravity Lagrangian is given by

规范超引力拉格朗日的玻色子部分为

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \mathcal{F}^{ab}_{\mu\nu}\mathcal{F}^{\mu\nu}_{ab} - \frac{1}{6}\mathcal{H}_{\mu\nu\rho ab}\mathcal{H}^{\mu\nu\rho}{}_{ab} - \frac{1}{2}P_{\mu ab}{}^{cd}P^{\mu}{}_{cd}{}^{ab} + e^{-1}\mathcal{L}_{\text{top}}$$

$$+\frac{1}{128}g^2\left(15B^2 + 2C^{ab}C_{ab} - 2B^{ab}{}_{cd}B^{cd}{}_{ab} - 2C^{[ab]}{}_{(cd)}C_{[ab]}{}^{(cd)}\right),$$

(51)

where $\mathcal{H}_{\mu\nu\rho ab} = \mathcal{V}^M{}_{ab}\mathcal{H}_{\mu\nu\rho M}$, and $\mathcal{L}_{\text{top}}$ has a complicated form provided explicitly in [94], with the field $C_{\mu\nu\rho}{}^M$ occurring only under the projection with Y_{MN} . Even though $\mathcal{L}_{\text{top}}$ is very complicated, its general variation is simple, and it is given in [94, eq. 3.16], where the fermionic terms and the supertransformations can be found as well. In particular the supertransformations of the fermions are given by

其中 $\mathcal{H}_{\mu\nu\rho ab} = \mathcal{V}^M{}_{ab}\mathcal{H}_{\mu\nu\rho M}$ ，且 $\mathcal{L}_{\text{top}}$ 形式复杂，文献 [94] 已给出其显式形式，其中场 $C_{\mu\nu\rho}{}^M$ 仅在 Y_{MN} 投影下出现。尽管 $\mathcal{L}_{\text{top}}$ 十分复杂，但其一般变分形式简单，结果参见 [94, 式 3.16]，费米项和超变换也可在该文献中找到。特别地，费米子的超变换由下式给出

$$\delta\psi^a_\mu = D_\mu\varepsilon^a + \frac{1}{5\sqrt{2}}\mathcal{F}^{ab}_{\mu\nu}\left(\gamma^{\nu\rho}{}_\mu + 8\gamma^\nu\delta^\rho_\mu\right)\varepsilon_b$$

$$+\frac{1}{15}\mathcal{H}^{ab}_{\nu\rho\lambda}\left(\gamma^{\nu\rho\lambda}{}_\mu + \frac{9}{2}\gamma^{\nu\rho}\delta^\lambda_\mu\right)\varepsilon_b + gA^{ab}_1\varepsilon_b,$$

$$\delta\chi^{abc} = -2P_\mu{}^{cdab}\gamma^\mu\varepsilon_d - \sqrt{2}\gamma^{\mu\nu}\left(\mathcal{F}^{c[a}_{\mu\nu}\varepsilon^{b]} + \frac{2}{5}\Omega^{a(b}\mathcal{F}^{c)d}_{\mu\nu}\varepsilon_d\right) \tag{52}$$

$$-\frac{1}{6}\gamma^{\mu\nu\rho}\left(\mathcal{H}^{[ab]}_{\mu\nu\rho}\varepsilon^c + \frac{1}{5}\Omega^{ab}\mathcal{H}^{cd}_{\mu\nu\rho}\varepsilon_d + \frac{4}{5}\Omega^{c[a}\mathcal{H}^{b]d}_{\mu\nu\rho}\right)\varepsilon_d - gA^{d,abc}_2\varepsilon_d,$$

where

其中

$$A^{ab}_1 = -\frac{1}{4\sqrt{2}}\left(\frac{1}{4}B\Omega^{ab} + \frac{1}{5}C^{ab}\right),$$

$$A^{d,abc}_2 = \frac{1}{2\sqrt{2}}\left[C^{[ab](cd)} - B^{[ab][cd]} + \frac{1}{4}\left(C^{ab}\Omega^{cd} + \frac{1}{5}\Omega^{ab}C^{cd} + \frac{4}{5}\Omega^{c[a}C^{b]d}\right)\right].$$

(53)

The degrees of freedom are correctly $128_B + 128_F$, thanks to the (projected) duality equation. With fermion terms suppressed, this equation is given by [94]

由于 (投影) 对偶方程的存在，自由度被正确 $128_B + 128_F$ 。在忽略费米子项的情况下，该方程由文献 [94] 给出如下

$$Y_{MN}\left(\mathcal{V}_{ab}^{N}\mathcal{H}^{\mu\upsilon\rho ab}-\frac{1}{3!}\varepsilon^{\mu\upsilon\rho\sigma_1\ldots\sigma_4}\mathcal{G}_{\sigma_1\ldots\sigma_4}{}^{N}\right)=0, \tag{54}$$

which arises as a field equation of $C_{\mu\upsilon\rho}{}^{M}$. We refer the reader to [94] for more details and several examples of gaugings based on different choices of the embedding tensor.

它是 $C_{\mu\upsilon\rho}{}^{M}$ 的场方程。更多细节以及基于不同嵌入张量选择的若干规范例子，读者可参阅文献 [94]

Half-Maximal 7D Gauged Supergravity Coupled to Vector Multiplets

耦合矢量多重态的半极大 7 维规范超引力

The $7D$ maximal supergravity for the multiplet (41) admits a consistent truncation to half-maximal $7D$ multiplet with the field content $\left(e_\mu{}^r, \phi, B_{\mu\nu}, 3A_\mu; \psi_\mu^i, \chi^i\right)$, with $i = 1, 2$ labelling an $USp\,(2)$ doublet, coupled to three vector multiplets $\left(A_\mu, 3\phi; \lambda\right)$, and the scalars parametrizing the coset $GL\,(4, \mathbb{R})\,/SO\,(4)$. This model with its $SO\,(4) \subset GL\,(4, \mathbb{R})$ gauging was constructed by Noether procedure in [95]. A dual version of half-maximal supergravity containing a massless field $A_{\mu\upsilon\rho}$ was constructed in [96], and a two parameter deformation was found, one having to do with $SU\,(2)$ gauging, and the other one with a topological mass term. The bosonic part of the Lagrangian is given by [96][9]

对应多重态 (41) 的 $7D$ 极大超引力可以一致截断为场内容为 $\left(e_\mu{}^r, \phi, B_{\mu\nu}, 3A_\mu; \psi_\mu^i, \chi^i\right)$ 的半极大 $7D$ 多重态，其中 $i = 1, 2$ 标记一个 $USp\,(2)$ 二重态，该多重态耦合三个矢量多重态 $\left(A_\mu, 3\phi; \lambda\right)$ ，标量参数化陪集 $GL\,(4, \mathbb{R})\,/SO\,(4)$ 。这种带 $SO\,(4) \subset GL\,(4, \mathbb{R})$ 规范的模型通过诺特定理程序在文献 [95] 中构造完成。包含无质量场 $A_{\mu\upsilon\rho}$ 的半极大超引力对偶版本在文献 [96] 中构造完成，并且发现了该模型的双参数形变: 一个参数和 $SU\,(2)$ 规范相关，另一个和拓扑质量项相关。拉格朗日量的玻色子部分由 [96][9] 给出

$$\begin{aligned}\varepsilon^{-1}\mathcal{L} = &-\frac{1}{2}R-\frac{1}{48}\sigma^{-4}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}-\frac{1}{4}\sigma^{2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\\ &+\frac{i}{24\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}F_{\mu\nu\rho\sigma}\,\mathrm{tr}\left(X_{\kappa\lambda\tau}-\frac{2\sqrt{2}}{3}hA_{\kappa\lambda\tau}\right)\\ &+g^2\sigma^{-2}+8\sqrt{2}gh\sigma^{2}-16h^{2}\sigma^{8},\end{aligned} \tag{55}$$

where $\sigma = \exp\left(-\phi/\sqrt{5}\right)$, $\mathrm{tr}\,(FF) = F_i{}^jF_j{}^i$, $F_{\mu\upsilon\rho\sigma} = 4\partial_{[\mu}A_{\upsilon\rho\sigma]}$, $F_i{}^j = dA_i{}^j + ig(A\wedge A)_i{}^j$, and the Chern-Simons form

其中 $\sigma = \exp\left(-\phi/\sqrt{5}\right)$, $\operatorname{tr}(FF) = F_i{}^j F_j{}^i$, $F_{\mu\upsilon\rho\sigma} = 4\partial_{[\mu}A_{\upsilon\rho\sigma]}$, $F_i{}^j = dA_i{}^j + ig(A\wedge A)_i{}^j$，且陈-西蒙斯形式

$$X_{\mu\nu\rho} = \operatorname{tr}\left(A_{[\mu}\partial_\upsilon A_{\rho]} + \frac{2}{3}igA_{[\mu}A_\upsilon A_{\rho]}\right), \tag{56}$$

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[9] Typographical errors present in [96] are corrected in [97], and the $F\wedge X$ term is corrected here.

[9] 文献 [96] 中的印刷错误已在文献 [97] 中修正，此处也修正了 $F\wedge X$ 项。

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and g, h are arbitrary constants. The supertransformations of the fermions are

且 g, h 为任意常数。费米子的超变换为

$$\delta\psi_{\mu i} = D_\mu\varepsilon_i + \frac{i\sigma}{10\sqrt{2}}\left(\gamma_\mu{}^{\rho\sigma} - 8\delta_\mu^\rho\gamma^\sigma\right)F_{\rho\sigma i}{}^j\varepsilon_j \tag{57}$$

$$+\frac{1}{80\sqrt{2}}\sigma^{-2}\left(\gamma_\mu{}^{\upsilon_1\ldots\upsilon_4} - \frac{8}{3}\delta_\mu^{\upsilon_1}\gamma^{\upsilon_2\ldots\upsilon_4}\right)F_{\upsilon_1\ldots\upsilon_4} - \frac{1}{5}\left(\frac{1}{\sqrt{2}}g\sigma^{-1} + 2h\sigma^4\right)\gamma_\mu\varepsilon_i,$$

$$\delta\chi_i = \frac{1}{2}\partial_\mu\phi\gamma^\mu\varepsilon_i - \frac{i}{2\sqrt{10}}\sigma\gamma^{\mu\upsilon}F_{\mu\upsilon i}{}^j\varepsilon_j + \frac{1}{24\sqrt{10}}\sigma^{-2}\gamma^{\mu_1\ldots\mu_4}F_{\mu_1\ldots\mu_4}\varepsilon_i$$

$$+\frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{2}}g\sigma^{-1} - 8h\sigma^4\right)\gamma_\mu\varepsilon_i \tag{58}$$

where $D_\mu\varepsilon_i = \nabla_\mu\varepsilon_i + igA_{\mu i}{}^j\varepsilon_j$. It was shown in [97] that if $h/g < 0$ the potential has no critical points, and that if $h/g > 0$, then it has two extrema, one preserving the full $OSp\left(8^\star \mid 2\right)$ supersymmetry of the AdS_7 background, while the other breaks the supersymmetry completely [97].

其中 $D_\mu\varepsilon_i = \nabla_\mu\varepsilon_i + igA_{\mu i}{}^j\varepsilon_j$ 。文献 [97] 证明: 当 $h/g < 0$ 时，势不存在临界点；当 $h/g > 0$ 时，势存在两个极值，其中一个保持 AdS_7 背景的完整 $OSp\left(8^\star \mid 2\right)$ 超对称性，另一个则完全破缺超对称性 [97]。

The most general gauging of half-maximal supergravity coupled to n vector multiplets has not been carried out so far in the embedding tensor formalism to our best knowledge, though the special case of $n = 3$ has been investigated in that framework [98]. Nonetheless, the coupling to n vector multiplets and the gauging of $(n+3)$ parameter group $G \subset SO(n,3)$ were constructed by Noether procedure in [99], as we summarize below.

据我们所知，在嵌入张量形式体系中，耦合 n 个矢量多重态的半极大超引力的最一般规范至今尚未完成，不过特例 $n = 3$ 已在该框架下得到研究 [98]。尽管如此，耦合 n 个矢量多重态以及对 $(n+3)$ 参数群 $G \subset SO(n,3)$ 的规范已经在文献 [99] 中通过诺特定理程序构造完成，我们总结如下。

Combining n -copies of Maxwell multiplet consisting of fields $(A_\mu, 3\phi, \lambda)$ with the supergavity multiplet gives the field content

将 n n 个由场 $(A_\mu, 3\phi, \lambda)$ 组成的麦克斯韦多重态与超引力多重态结合，得到的场内容为

$$\{e_\mu{}^a, \phi, B_{\mu\nu}, \mathcal{V}_M{}^A, A_\mu^M; \psi_\mu, \chi^i, \lambda^{ai}\}, \tag{59}$$

where $M, A = 1, \ldots, n+3, a = 1, \ldots, n$, the scalar ϕ is the dilaton, the scalars $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^{ij}, \mathcal{V}_M{}^a)$, with $i, j = 1, 2$ labelling $USp(2)$ doublet and $\mathcal{V}_M{}^{ij}\Omega_{ij} = 0$, parametrize the $SO(n,3)/SO(n) \times SO(3)$ coset, and the fermions are symplectic Majorana. Writing $A_\mu^M = (A_\mu^m, A_\mu^\alpha)$ with $m = 1, 2, 3$ and $\alpha = 1, \ldots, n$, the vector fields A_μ^m belong to the supergravity multiplet. The $SO(n,3)$ invariant tensor $\eta = \mathrm{diag}(-1, -1, -1, +1, \ldots, +1)$ and the positive definite scalar matrix $\mathcal{M}$ are given by

其中 $M, A = 1, \ldots, n+3, a = 1, \ldots, n$ ，标量 ϕ 是 dilation，标量 $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^{ij}, \mathcal{V}_M{}^a)$ 满足: $i, j = 1, 2$ 标记 $USp(2)$ 二重态，$\mathcal{V}_M{}^{ij}\Omega_{ij} = 0$ 参数化了 $SO(n,3)/SO(n) \times SO(3)$ 陪集，费米子为辛马约拉纳费米子。记 $A_\mu^M = (A_\mu^m, A_\mu^\alpha)$ ，其中 $m = 1, 2, 3$ 和 $\alpha = 1, \ldots, n$ ，矢量场 A_μ^m 属于超引力多重态。$SO(n,3)$ 不变张量 $\eta = \mathrm{diag}(-1, -1, -1, +1, \ldots, +1)$ 与正定标量矩阵 $\mathcal{M}$ 由下式给出

$$\eta_{MN} = -V_I{}^i{}_j V_J{}^j{}_i + V_M{}^a V_N{}^a, \; \mathcal{M}_{MN} = V_I{}^i{}_j V_J{}^j{}_i + V_M{}^a V_N{}^a. \tag{60}$$

The inverse of $\mathcal{V}_M{}^A$ is $\mathcal{V}^M{}_A$, satisfying the relations $\mathcal{V}_M{}^i{}_j \mathcal{V}^{Mk}{}_\ell = -\delta^i_\ell \delta^k_j + \frac{1}{2}\delta^i_j \delta^k_\ell, \mathcal{V}_M{}^a \mathcal{V}^M{}_b = \delta^a_b$ and $\mathcal{V}_M{}^a \mathcal{V}^M{}_{ij} = 0$. Using the $(n+3)$ vector fields to gauge a $(n+3)$ parameter semisimple group $G \subset SO(n,3)$, we introduce the field strengths

$\mathcal{V}_M{}^A$ 的逆为 $\mathcal{V}^M{}_A$ ，满足关系 $\mathcal{V}_M{}^i{}_j \mathcal{V}^{Mk}{}_\ell = -\delta^i_\ell \delta^k_j + \frac{1}{2}\delta^i_j \delta^k_\ell, \mathcal{V}_M{}^a \mathcal{V}^M{}_b = \delta^a_b$ 和 $\mathcal{V}_M{}^a \mathcal{V}^M{}_{ij} = 0$ 。利用 $(n+3)$ 个矢量场对带 $(n+3)$ 个参数的半单群 $G \subset SO(n,3)$ 做定规范，我们引入场强:

$$\mathcal{F}_{\mu\nu}^M = 2\partial_{[\mu} A_{\nu]}^M + f_{NP}{}^M A_\mu^N A_\nu^P,$$

$$\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} - \frac{3}{\sqrt{2}} \eta_{MN} \mathcal{F}_{[\mu\nu}^M A_{\rho]}^N - \frac{1}{\sqrt{2}} f_{MN}{}^P A_{[\mu}^M A_\nu^N A_{\rho]P}, \tag{61}$$

with $f_{MN}{}^P$ representing structure constants of G . We also need the coset currents and $SO(n) \times SO(3)$ composite connections

其中 $f_{MN}{}^P$ 代表 G 的结构常数。我们还需要陪集流与 $SO(n) \times SO(3)$ 复合联络

$$\mathcal{P}_{\mu a}{}^i{}_j = \mathcal{V}_a^M (\partial_\mu \delta_M^N + f_{MK}{}^N A_\mu^K) \mathcal{V}_N{}^i{}_j$$

$$\mathcal{Q}_\mu{}^i{}_j = \mathcal{V}^{Mi}{}_k (\partial_\mu \delta_M^N + f_{MK}{}^N A_\mu^K) \mathcal{V}_N{}^k{}_j,$$

$$\mathcal{Q}_{\mu ab} = \mathcal{V}^M{}_a (\partial_\mu \delta_M^N + f_{MK}{}^N A_\mu^K) \mathcal{V}_{Nb} \tag{62}$$

Using these building blocks, the Lagrangian takes the form [95]

利用这些构造块，拉格朗日量形式为 [95]

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{4}e^{\phi}\mathcal{M}_{MN}\mathcal{F}^M_{\mu\nu}\mathcal{F}^{\mu\nu N} - \frac{1}{12}e^{2\varphi}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} + \frac{5}{8}\partial_\mu\phi\partial^\mu\phi$$

$$-\frac{1}{2}\mathcal{P}^{ai}_{\mu}{}_j\mathcal{P}^{\mu}{}_{ai}{}^j - \frac{1}{4}e^{-\phi}\left(C^{ai}{}_jC_{ai}{}^j - \frac{1}{9}C^2\right), \tag{63}$$

where

其中

$$C = if_{MN}{}^P\mathcal{V}^{Mi}{}_k\mathcal{V}^{Nj}{}_i\mathcal{V}_P{}^k{}_j,\; C^{ai}{}_j = if_{MN}{}^P\mathcal{V}^{Mi}{}_k\mathcal{V}^{Nk}{}_j\mathcal{V}_P{}^a, \tag{64}$$

and the supertransformations of the fermions are

费米子的超变换为

$$\delta\psi^i_\mu = 2D_\mu\varepsilon^i - \frac{1}{60}e^{\phi}\left(\gamma_\mu\gamma^{\rho\sigma\tau} + 5\gamma^{\rho\sigma\tau}\gamma_\mu\right)\mathcal{H}_{\rho\sigma\tau}\varepsilon$$

$$+\frac{i}{10\sqrt{2}}e^{\phi/2}\left(3\gamma_\mu\gamma^{\rho\sigma} - 5\gamma^{\rho\sigma}\gamma_\mu\right)\mathcal{F}^M_{\rho\sigma}\mathcal{V}_M{}^i{}_j\varepsilon^j - \frac{\sqrt{2}}{30}e^{-\phi/2}C\gamma_\mu\varepsilon^i,$$

$$\delta\chi^i = -\frac{1}{2}\gamma^\mu\partial_\mu\phi + \frac{1}{5\sqrt{2}}e^{\phi/2}\gamma^{\mu\nu}\mathcal{F}^M_{\mu\nu}\mathcal{V}_M{}^i{}_j\varepsilon^j$$

$$-\frac{1}{15\sqrt{2}}e^{\phi}\gamma^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho}\varepsilon^i + \frac{\sqrt{2}}{30}e^{-\phi/2}C\varepsilon^i,$$

$$\delta\lambda^{ai} = -\frac{1}{2}e^{\phi/2}\gamma^{\mu\nu}\mathcal{F}^M_{\mu\nu}\mathcal{V}_M{}^a\varepsilon^i - \sqrt{2}i\gamma^\mu\mathcal{P}_\mu{}^{ai}{}_j\varepsilon^j + ie^{-\phi/2}C^{ai}{}_j\varepsilon^j. \tag{65}$$

Turning to the nature of the gauge group G with the structure constants $f_{MN}{}^P$, it has been observed that it has the form $G_0 \times H \subset SO(3,n)$ where G_0 is one of the following [100]

讨论带结构常数 $f_{MN}{}^P$ 的规范群 G 的性质，已知其形式为 $G_0 \times H \subset SO(3,n)$，其中 G_0 为以下之一 [100]

$$SO(2,1),\; SO(3,1),\; SO(2,2),\; SO(2,2)\times SO(2,1),\; SL(3,R),\; SO$$

(66)

and H is $(n+3-\dim G_0)$ dimensional gauge group. The question of whether this provides the most general gaugings, the possibility of further deformations, and the question of which one of these theories can be embedded in $10D$ or $11D$ supergravities are remain to be investigated. For a reduction of $N=1, 10D$ heterotic supergravity on a group manifold, in particular, see [101]. The supersymmetric double field theory and exceptional field theory approaches, both briefly reviewed in section ”Supergravities in Extended Geometry

Framework", may also shed light on the embedding questions. In particular, it is worth noting that the Lagrangian (38) agrees with the result in [89, eq. (3.43)] obtained in a certain reduction scheme of the (gauged) double field theory.

且 H 是 $(n+3-\dim G_0)$ 维规范群。这是否涵盖了所有最一般的规范定法，是否存在进一步形变，以及这些理论中哪些可以嵌入 $10D$ 或 $11D$ 超引力，这些问题仍有待研究。特别地，关于 $N=1, 10D$ 杂弦超引力在群流形上的约化，参见文献 [101]。"扩展几何框架下的超引力"一节简要回顾了超对称双重场论方法和例外场论方法，它们也可为嵌入问题提供启发。特别值得注意的是，拉格朗日量 (38) 与文献 [89, 式 (3.43)] 在 (定规范) 双重场论的某个约化方案中得到的结果一致。

D = 6

(2, 2) Gauged Supergravity in 6D

六维 (2, 2) 规范超引力

The ungauged $(2,2)$ theory was constructed in [102] and it has the duality symmetry $SO(5,5)$. Using the embedding tensor formalism, the general gaugings of a subgroup $G \subset SO(5,5)$ that utilizes the 16 vector fields of the supergravity multiplet were carried out in [103], as we summarize next. The maximal gauged supergravity is built out of the following multiplet of fields

未规范 $(2,2)$ 理论由文献 [102] 构造，具有对偶对称性 $SO(5,5)$ 。利用嵌入张量形式论，文献 [103] 完成了对利用超引力多重态 16 个矢量场的子群 $G \subset SO(5,5)$ 的一般规范，我们将在下文总结。最大规范超引力由以下场多重态构造

$$\left(e_\mu{}^r, V_A{}^{\alpha\dot\alpha}, A_\mu^A, B_{\mu\upsilon M}, C_{\mu\upsilon\rho A}; \psi_{+\mu\alpha}, \psi_{-\mu\dot\alpha}, \chi_{+a\dot\alpha}, \chi_{-\dot a\alpha}\right), \tag{67}$$

where $M=1,\ldots,10$ labels the fundamental and $A=1,\ldots,16$ labels the spinor representation of the duality group $SO(5,5)$. The indices $\alpha,\dot\alpha=1,\ldots 4$ label the spinor representations of $SO(5)\times SO(5)$, and $V_A^{\alpha\dot\alpha}$ is the $SO(5,5)/SO(5)\times SO(5)$ coset representative. The spinors are symplectic Majorana-Weyl, $a,\dot a = 1,\ldots 5$ are the vectors indices of $SO(5)\times SO(5)$ and $\pm$ refer to chirality under γ_7 . The two-form potential $B_{\mu\upsilon M}=\left(B_{\mu\upsilon m}, B_{\mu\upsilon}{}^m\right)$ consists of the electric and magnetic two-forms B_m and B^m transforming as 5 and 5′ of $GL(5)$, respectively, and combining into10of $SO(5,5)$. The three-form $C_{\mu\upsilon\rho A}$ is introduced as on-shell dual of the vector fields A_μ^A . Their properly covariantized field strengths will be related to each other via a duality relation, which arises, under a projection, as an equation of motion.

其中 $M=1,\ldots,10$ 标记对偶群 $SO(5,5)$ 的基础表示,$A=1,\ldots,16$ 标记其旋量表示。指标 $\alpha,\dot\alpha=1,\ldots 4$ 标记 $SO(5)\times SO(5)$ 的旋量表示，$V_A^{\alpha\dot\alpha}$ 是陪集 $SO(5,5)/SO(5)\times SO(5)$ 的代表元。旋量为辛马约拉纳-外尔旋量，$a,\dot a=1,\ldots 5$ 是 $SO(5)\times SO(5)$ 的矢量指标，$\pm$ 对应 γ_7 下的手征性。二形式势 $B_{\mu\upsilon M}=\left(B_{\mu\upsilon m}, B_{\mu\upsilon}{}^m\right)$ 由电二形式 B_m 和磁二形式 B^m 组成，二者分别按 $GL(5)$ 的 5 维表示和 5′ 变换，组合后构成 $SO(5,5)$ 的 10 维表示。引入三形式 $C_{\mu\upsilon\rho A}$ 作为矢量场 A_μ^A 的壳上对偶。它们经恰当协变化的场强通过对偶关系相互联系，该对偶关系是投影下作为运动方程出现的。

All of the 16 gauge fields, or a subset of them, may be used to gauge a suitable subgroup of $S0(5,5)$. This gauging is encoded in a real embedding tensor $\theta_A{}^{MN}=\theta_A{}^{[MN]}$, which determines the generators of the gauge group G among the $SO(5,5)$ generators t_{MN} and thus the covariant derivative

全部 16 个规范场 (或其中一个子集) 可用于规范 $S0(5,5)$ 的一个合适子群。该规范由实嵌入张量 ${\theta_A}^{MN} = {\theta_A}^{[MN]}$ 编码，它确定了规范群 G 的生成元在 $SO(5,5)$ 生成元 t_{MN} 中的位置，进而给出协变导数

$$X_A = {\theta_A}^{MN} t_{MN},\ D_\mu = \nabla_\mu - g A_\mu^A X_A. \tag{68}$$

Supersymmetry imposes a linear constraint on the embedding tensor such that in the product $\mathbf{16} \otimes \mathbf{45}$, only the representations $\mathbf{144}_c$ survives, and therefore it can be parametrized as

超对称对嵌入张量施加了一个线性约束: 在乘积 $\mathbf{16} \otimes \mathbf{45}$ 中，仅表示 $\mathbf{144}_c$ 保留，因此嵌入张量可参数化为

$$\theta_A^{MN} = -\theta^{B[M}\gamma_{BA}^{N]} \tag{69}$$

where γ_{AB}^M are chirally projected gamma-matrices of $SO(5,5)$. The closure of the gauge algebra imposes the quadratic constraints

其中 γ_{AB}^M 是 $SO(5,5)$ 的手征投影伽马矩阵。规范代数的封闭性要求满足二次约束

$$\theta^{AM}\theta^{BN}\eta_{MN} = 0,\ \theta^{AM}\theta^{B[N}\gamma_{AB}^{P]} = 0, \tag{70}$$

where η_{MN} is the invariant tensor of $SO(5,5)$. The covariant field strengths are given by (from here on we absorb g into the definition of the embedding tensor)

其中 η_{MN} 是 $SO(5,5)$ 的不变张量。协变场强由下式给出 (下文中我们将 g 吸收到嵌入张量的定义中)

$$\mathcal{F}_{\mu\nu}^A \equiv 2\partial_{[\mu}{A_{\nu]}}^A + {X_{[BC]}}^A {A_\mu}^B {A_\nu}^C - \sqrt{2}\theta^{AM} B_{\mu\nu M},$$

$$\mathcal{H}_{\mu\nu\rho M} \equiv 3D_{[\mu}B_{\upsilon\rho]M} + 3\sqrt{2}(\gamma_M)_{AB} A_{[\mu}{}^A \left(\partial_\upsilon {A_{\rho]}}^B + \frac{1}{3}{X_{[CD]}}^B {A_\upsilon}^C {A_{\rho]}}^D\right)$$

$$-\sqrt{2}\theta^{AN}\eta_{MN}C_{\mu\upsilon\rho A}$$

$$\mathcal{G}_{\mu\upsilon\rho\lambda A} \equiv 4D_{[\mu}C_{\upsilon\rho\lambda]A} - \left(\gamma^M\right)_{AB}\left(6\sqrt{2}B_{[\mu\upsilon M]}\mathcal{F}_{\rho\lambda]}^B + 6\theta^{BN}B_{[\mu\upsilon M}B_{\rho\lambda]N}\right. \tag{71}$$

$$\left.+8(\gamma_M)_{CD}A_{[\mu}^B A_\nu^C \partial_\rho A_{\lambda]}^D + 2(\gamma_M)_{CF}{X_{DE}}^F A_{[\mu}{}^B {A_\nu}^C {A_\rho}^D {A_{\lambda]}}^E\right),$$

where ${X_{AB}}^C = \left(\gamma^M\theta^N\right)_A (\gamma_{MN})_B{}^C$. Next, we define the coset currents and the $SO(5)$ composite connection in terms of the $SO(5,5)$ valued 16×16 matrix $V_\alpha^{\alpha\dot\alpha}$ as

其中 $X_{AB}{}^{C} = \left(\gamma^{M}\theta^{N}\right)_{A}(\gamma_{MN})_{B}{}^{C}$。接下来我们将陪流和 $SO(5)$ 复合联络用取值于 $SO(5,5)$ 的 16×16 矩阵 $V_{\alpha}^{\alpha\dot{\alpha}}$ 定义为

$$P_{\mu}^{a\dot{a}} = \frac{1}{4}\bar{V}\gamma^{a}\gamma^{\dot{a}}\mathcal{D}_{\mu}V,\ Q_{\mu}^{ab} = \frac{1}{8}\bar{V}\gamma^{ab}\mathcal{D}_{\mu}V, \tag{72}$$

where $\mathcal{D}_{\mu}V = D_{\mu}(Q)V - \left(\bar{A}_{\mu}\gamma^{M}\theta^{N}\right)\gamma_{MN}V$. To construct the T -tensors, we first introduce 10×10 scalar matrix $\mathcal{V}$ that is an element of $SO(5,5)$. For convenience explained in detail in [103], we choose $\mathcal{V}$ such that it satisfies $\mathcal{V}^{T}\eta\mathcal{V} = \eta_{\text{diag}}$ where $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\eta_{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Writing

其中 $\mathcal{D}_{\mu}V = D_{\mu}(Q)V - \left(\bar{A}_{\mu}\gamma^{M}\theta^{N}\right)\gamma_{MN}V$。为构造 T 张量，我们首先引入 10×10 标量矩阵 $\mathcal{V}$，它是 $SO(5,5)$ 的一个元素。为方便起见 (详见文献 [103])，我们选取满足 $\mathcal{V}^{T}\eta\mathcal{V} = \eta_{\text{diag}}$ 的 $\mathcal{V}$，其中 $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$，且 $\eta_{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$。写为

$$\mathcal{V}_{M}^{A} = \begin{pmatrix} \mathcal{V}_{m}^{a} & \mathcal{V}_{m}^{\dot{a}} \\ \mathcal{V}^{ma} & \mathcal{V}^{m\dot{a}} \end{pmatrix} \tag{73}$$

we find the relations

我们得到关系

$$\mathcal{V}_{M}{}^{a} = \frac{1}{16}\bar{V}\gamma_{M}\gamma^{a}V,\ \mathcal{V}_{M}{}^{\dot{a}} = -\frac{1}{16}\bar{V}\gamma_{M}\gamma^{\dot{a}}V. \tag{74}$$

The T -tensors are then defined as

随后 T 张量定义为

$$T^{a} = \mathcal{V}_{M}{}^{a}\theta^{AM}V_{A},\ T^{\dot{a}} = -\mathcal{V}_{M}{}^{\dot{a}}\theta^{AM}V_{A}. \tag{75}$$

Note that the $\alpha\dot{\alpha}$ indices of $V_{A}^{\alpha\dot{\alpha}}$ are suppressed. Finally, we define the following hyper-matrix which is key for writing the appropriate kinetic term for the two-form potential:

注意，$V_{A}^{\alpha\dot{\alpha}}$ 的 $\alpha\dot{\alpha}$ 指标已省略。最后，我们定义如下超矩阵，它是写出二形式势恰当动能项的关键:

$$K^{mn} = \mathcal{V}^{ma}(\mathcal{V}_{n}^{a})^{-1}P_{+} - \mathcal{V}^{m\dot{a}}(\mathcal{V}_{n}^{\dot{a}})^{-1}P_{-}, \tag{76}$$

where $P_{\pm} = \frac{1}{2}(1 \pm j)$ and j acts on a given the three-form as $j\omega = \widetilde{\omega}$ with the definition $\widetilde{\omega} = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma\lambda\tau}\omega^{\sigma\lambda\tau}$. The bosonic part of the Lagrangian takes the form [103]

其中 $P_{\pm} = \frac{1}{2}(1 \pm j)$，且 j 按照 $j\omega = \widetilde{\omega}$ 作用于给定三形式，定义为 $\widetilde{\omega} = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma\lambda\tau}\omega^{\sigma\lambda\tau}$。拉格朗日量的玻色子部分形式为 [103]

$$e^{-1}\mathcal{L}_{B} = \frac{1}{4}R - \frac{1}{12}\mathcal{H}_{m}\cdot K^{mn}\mathcal{H}_{n} - \frac{1}{4}\mathcal{M}_{AB}\mathcal{F}_{\mu\nu}^{A}\mathcal{F}^{\mu\nu B} - \frac{1}{16}\mathcal{P}_{\mu}^{a\dot{a}}\mathcal{P}_{a\dot{a}}^{\mu}$$

$$+\left(\mathrm{tr}\,T^a\widetilde{T}^a-\frac{1}{2}\,\mathrm{tr}\,T\widetilde{T}\right)+e^{-1}\mathcal{L}_{\,\mathrm{top}}\,, \tag{77}$$

where " $\sim$ ", denotes transposition, $M_{AB}=V_A^{\alpha\dot{\alpha}}V_{B\alpha\bar{\delta}}$ and $\mathcal{L}_{\,\mathrm{top}}$ is the topological part of the Lagrangian given explicitly in [103]. It is a complicated expression but its general variation takes a simple form, which makes it straightforward to derive the following equations of motion for the three-form potential C_A and the "magnetic" two-form potentials B_M :

其中" $\sim$ " 表示转置，$M_{AB}=V_A^{\alpha\dot{\alpha}}V_{B\alpha\bar{\delta}}$ ，$\mathcal{L}_{\,\mathrm{top}}$ 是拉格朗日量的拓扑项，其显式形式可见 [103]。它的表达式较为复杂，但一般变分形式十分简单，因此可以直接推导出三形式势 C_A 和" 磁性" 二形式势 B_M 的运动方程:

$$\theta_m^A\left(\mathcal{H}^m-jK^{mn}\mathcal{H}_n\right)=0,\;\theta_m^A\left(\mathcal{G}_{\mu\upsilon\rho\sigma A}+\frac{1}{2}\varepsilon_{\mu\upsilon\rho\sigma\lambda\tau}\mathcal{M}_{AB}\mathcal{F}^{\lambda\tau B}\right)=0.$$

(78)

These first equation furnishes a duality relation between the electric and magnetic two forms, combined as $\mathcal{H}_M=(\mathcal{H}_m,\mathcal{H}^m)$, and the second equation between the three-form potentials and the vector fields. Finally, the supersymmetry transformations of the fermionic field are given by [103]

第一个方程给出了组合为 $\mathcal{H}_M=(\mathcal{H}_m,\mathcal{H}^m)$ 的电性和磁性二形式之间的对偶关系，第二个方程给出三形式势与矢量场之间的对偶关系。最后，费米子场的超对称变换由下式给出 [103]

$$\delta\psi_{\mu+}=\mathcal{D}_\mu\varepsilon_+-\frac{1}{24}\mathcal{H}^a_{\rho\sigma\kappa}\gamma^a\gamma^{\rho\sigma\kappa}\gamma_\mu\varepsilon_++\frac{1}{8}\left(\gamma_\mu{}^{\nu\rho}-6\delta_\mu^\nu\gamma^\rho\right)\mathcal{F}^A_{\nu\rho}V_A\varepsilon_-+\frac{1}{4}\gamma_\mu T\varepsilon_-,$$

$$\delta\psi_{\mu-}=\mathcal{D}_\mu\varepsilon_--\frac{1}{24}\mathcal{H}^{\dot{a}}_{\rho\sigma\kappa}\gamma^{\dot{a}}\gamma^{\rho\sigma\kappa}\gamma_\mu\varepsilon_-+\frac{1}{8}\left(\gamma_\mu{}^{\nu\rho}-6\delta_\mu^\nu\gamma^\rho\right)\mathcal{F}^A_{\nu\rho}\widetilde{V}_A\varepsilon_+-\frac{1}{4}\gamma_\mu\widetilde{T}\varepsilon_+$$

$$\delta\chi^{\dot{a}}=\frac{1}{4}\mathcal{P}^{a\dot{a}}_\mu\gamma^a\gamma^\mu\varepsilon+\frac{1}{12}\mathcal{H}^{\dot{a}}_{\mu\nu\rho}\gamma^{\mu\upsilon\rho}\varepsilon+\frac{1}{4}\mathcal{F}^A_{\mu\nu}V_A\gamma^{\dot{a}}\gamma^{\mu\upsilon}\varepsilon+2T^{\dot{a}}\varepsilon+\frac{1}{2}T\gamma^{\dot{a}}\varepsilon,$$

$$\delta\chi^a=\frac{1}{4}\mathcal{P}^{a\dot{a}}_\mu\gamma^{\dot{a}}\gamma^\mu\varepsilon+\frac{1}{12}\mathcal{H}^a_{\mu\nu\rho}\gamma^{\mu\upsilon\rho}\varepsilon+\frac{1}{4}\mathcal{F}^A_{\mu\nu}\widetilde{V}_A\gamma^a\gamma^{\mu\upsilon}\varepsilon+2\widetilde{T}^a\varepsilon-\frac{1}{2}\widetilde{T}\gamma^a\varepsilon.$$

(79)

Possible solutions of the embedding constraint (70) and the identification of the resulting theories were provided in [103]. In particular, decomposing θ^{AM} under $GL(5)\subset SO(5,5)$ allows identification of possible $7D$ origins, as well as a possible origin in $11D$, in which context $GL(5)$ is associated with the five-torus on which the reduction is performed. Gaugings of half-maximal $6D$ supergravity coupled to 4 vector multiplets (see the subsection below), which has the duality group $R^+\times SO(4,4)$ can also be obtained by decomposing θ^{AM} under this duality group and performing a truncation to $N=(1,1)$ supersymmetry [104]. For the details of the classifications of gaugings along these lines, see [103].

嵌入约束 (70) 的可能解和由此得到的理论分类已在文献 [103] 中给出。特别地，将 θ^{AM} 按 $GL(5)\subset SO(5,5)$ 分解后，我们可以确定可能的 $7D$ 起源，以及它在 $11D$ 中的一种可能起源，在该背景下 $GL(5)$ 与约化所依赖的五圆环相关联。耦合 4 个矢量多重态的半最大 $6D$ 超引力 (见下述小节) 其对偶群为 $R^+\times SO(4,4)$ ，也可以通过将 θ^{AM} 在该对偶群下分解，并做截断得到 $N=(1,1)$ 超对称来得到 [104]。关于沿这一思路对规范场进行分类的细节，参见文献 [103]。

(1, 1) Supergravity Coupled to Vector Multiplets in 6D

六维 (1,1) 超引力与矢量多重态耦合

Pure $(1,1)$, $6D$ supergravity is based on the Poincaré superalgebra $F(4)$. Its $SU(2)$ gauged version coupled to a single vector multiplet was obtained in [105] by circle reduction of $SU(2)$ gauged half-maximal $7D$ supergravity with a topological mass term [96], summarized above in section "Half-Maximal 7D Gauged Supergravity Coupled to Vector Multiplets". [10] The $6D$ theories obtained in this way have no stable ground states with maximal spacetime symmetry. It was shown [107] that for $g>0, h=0$ with $SU(2)_{\text{diag}} \subset SO(4)_R \approx SU(2)\times SU(2)$ gauged, the theory can be generalized by introducing a mass parameter m for the two-form tensor $B_{\mu\nu}$, just as in type IIA supergravity in $10D$. (The parameters (g,h,m) are defined in section "Half-Maximal 7D Gauged Supergravity Coupled to Vector Multiplets".) Furthermore it was shown [107] that this generalized theory for $g>0, m>0$ admits a ground state which exhibits the full AdS supersymmetry $F(4)$.

纯 $(1,1)$, $6D$ 超引力基于庞加莱超代数 $F(4)$。其耦合单个矢量多重态的 $SU(2)$ 定标版本由文献 [105] 通过对带有拓扑质量项的 $SU(2)$ 定标半最大 $7D$ 超引力 [96] 做圆约化得到，我们已经在前面“半最大七维定标超引力与矢量多重态耦合”一节做了总结。[10] 通过这种方式得到的 $6D$ 理论不存在具有最大时空对称性的稳定基态。文献 [107] 表明，对于带 $SU(2)_{\text{diag}} \subset SO(4)_R \approx SU(2)\times SU(2)$ 定标的 $g>0, h=0$，该理论可以通过引入二形式张量 $B_{\mu\nu}$ 的质量参数 m 来推广，这和 $10D$ 中 IIA 型超引力的情况一致。(参数 (g,h,m) 已在“半最大七维定标超引力与矢量多重态耦合”一节中定义。) 此外文献 [107] 还证明，这个推广后的理论对 $g>0, m>0$ 允许一个拥有完整 AdS 超对称性 $F(4)$ 的基态。

The coupling to n vector multiplets, which has the duality group $R^+ \times SO(n,4)/SO(n)\times SO(4)$, and gauging of $SU(2)_{\text{diag}} \times G$, where G is an n dimensional subgroup of $SO(n)$, was carried out in [108]. More general gauging inherited from the truncation of the maximal $N=(2,2)$ supergravity described in the previous subsection to half-maximal $N=(1,1)$ supergravity, the resulting potential, and its extrema were studied in [104]. However, the most general $6D, N=(1,1)$ theory in the embedding tensor framework remains to be spelled out. Here we shall summarize the bosonic action and supertransformations obtained in [108], bearing in mind that it provides a particular gauging.

耦合 n 个矢量多重态 (对偶群为 $R^+ \times SO(n,4)/SO(n)\times SO(4)$)，以及 $SU(2)_{\text{diag}} \times G$ 的定标 (其中 G 是 $SO(n)$ 的 n 维子群) 的工作已在文献 [108] 中完成。文献 [104] 研究了从上一小节描述的最大 $N=(2,2)$ 超引力截断得到的更一般定标、得到的半最大 $N=(1,1)$ 超引力、相应势能及其极值。然而，最一般的 $6D, N=(1,1)$ 理论在嵌入张量框架下的具体形式仍有待阐明。我们这里将总结文献 [108] 中得到的玻色子作用量和超变换，请注意这是一种特殊定标。

Combining n -copies of Maxwell multiplet consisting of fields $\left(A_\mu, 4\phi, \lambda\right)$ with the supergavity multiplet gives the field content

将由场 $\left(A_\mu, 4\phi, \lambda\right)$ 构成的 n 份麦克斯韦多重态和超引力多重态结合，得到的场内容为

$$\left\{e_\mu{}^r, \varphi, B_{\mu\nu}, \mathcal{V}_M{}^A, A_\mu^M; \psi_\mu^i, \chi^i, \lambda^{ai}\right\}, \tag{80}$$

where $M, A = 1,\ldots,n+4, a = 1,\ldots,n$, the scalar ϕ is dilaton, and $\mathcal{V}_M{}^A$ is the $SO(n,4)/SO(n)\times$

$SO(4)$ coset representative parametrized by $4n$ scalars. The fermions are symplectic Majorana, with $i=1,2$ labeling an $SU(2)$ doublet. Denoting $SU(2)$ generators by T^I, note that $T^I P_\pm$, where $P_\pm = \frac{1}{2}(1 \pm \gamma_7)$, act independently on spinors, yielding the isomorphism group $SU(2) \times SU(2)$ of the supersymmetry algebra. Writing $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a)\,(m=1,\ldots,4)$, the vectors A_μ^m belong to the supergravity multiplet. In a $1+3$ split we have, $\mathcal{V}_M{}^m(\sigma^m)_{ij} = \mathcal{V}_M{}^0\varepsilon_{ij} + \mathcal{V}_M{}^I(\sigma^I)_{ij}$, with $I=1,2,3$. In gauging the group $G \subset SO(2)_{\text{diag}} \times SO(n)$, we need the scalar currents [11]

其中 $M,A=1,\ldots,n+4, a=1,\ldots,n$，标量 ϕ 是 dilaton(dilaton 场/伸缩子)，$\mathcal{V}_M{}^A$ 是由 $4n$ 个标量参数化的 $SO(n,4)/SO(n)\times SO(4)$ 陪集代表元。费米子为辛马约拉纳费米子，其中 $i=1,2$ 标记一个 $SU(2)$ 二重态。将 $SU(2)$ 的生成元记为 T^I，注意 $T^I P_\pm$ (其中 $P_\pm = \frac{1}{2}(1\pm\gamma_7)$) 独立作用于旋量，得到超对称代数的同构群 $SU(2)\times SU(2)$。记 $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a)\,(m=1,\ldots,4)$ 后，矢量 A_μ^m 属于超引力多重态。在 $1+3$ 分解下我们得到 $\mathcal{V}_M{}^m(\sigma^m)_{ij} = \mathcal{V}_M{}^0\varepsilon_{ij} + \mathcal{V}_M{}^I(\sigma^I)_{ij}$，且满足 $I=1,2,3$。对群 $G\subset SO(2)_{\text{diag}} \times SO(n)$ 做规范作用时，我们需要标量流 [11]

$$\mathcal{P}_\mu^{a0} = \mathcal{V}^{Ma} D_\mu \mathcal{V}_M{}^0,\ \mathcal{P}_\mu^{aI} = \mathcal{V}^{Ma} D_\mu \mathcal{V}_M{}^I,$$

$$\mathcal{Q}_\mu^{IJ} = \mathcal{V}^{M[I} D_\mu \mathcal{V}_M{}^{J]},\ \mathcal{Q}_\mu^{ab} = \mathcal{V}^{M[a} D_\mu \mathcal{V}_M{}^{b]}, \tag{81}$$

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[10] A generalized dimensional reduction of half-maximal $7D$ supergravity on a circle in which the scale and trombone symmetries are utilized was obtained in [106] giving rise to a (1, 1) theory in $6D$ with four abelian vectors, and whose equations of motion cannot be derived from a Lagrangian.

[10] A 利用标度对称性与长号对称性对半极大 $7D$ 超引力做广义圆维约化的工作已在文献 [106] 中完成，得到了 $6D$ 维下含四个阿贝尔矢量的 (1,1) 理论，该理论的运动方程无法从拉格朗日量导出。

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where

其中

$$D_\mu \mathcal{V}_M{}^A = \partial_\mu \mathcal{V}_M{}^A + f_{MP}{}^N A_\mu^P \mathcal{V}_N{}^A,\ f_{MNP} = \{g\varepsilon_{IJK}, g' f_{abc}\}, \tag{82}$$

where f_{abc} are the structure constants of n -dimensional subgroup of $SO(n)$. Further needed definitions are those of the "boosted structure constants,"

其中 f_{abc} 是 n 维 $SO(n)$ 子群的结构常数。进一步还需要定义“提升结构常数”，

$$C = f_{MNP}\mathcal{V}_I^M \mathcal{V}_J^N \mathcal{V}_K^P \varepsilon^{IJK},\ C^I = f_{MNP}\mathcal{V}_J^M \mathcal{V}_K^N \mathcal{V}_0^P \varepsilon^{IJK},$$

$$C_{1a}{}^I = f_{MNP}\mathcal{V}^M{}_a \mathcal{V}_J^N \mathcal{V}_K^P \varepsilon^{IJK},\ C_{2aI} = f_{MNP}\mathcal{V}_0^M \mathcal{V}_I^N \mathcal{V}^P{}_a. \tag{83}$$

The bosonic part of the Lagrangian is given by [108]

拉格朗日量的玻色子部分由文献 [108] 给出:

$$
\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{4}R - \frac{1}{8}e^{-2\varphi}\mathcal{M}_{MN}\mathcal{F}^M_{\mu\upsilon}\mathcal{F}^{\mu\upsilon N} + \frac{3}{64}e^{4\phi}H_{\mu\upsilon\rho}H^{\mu\upsilon\rho} \\
& +\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\mathcal{P}^{a0}_\mu\mathcal{P}^\mu_{a0} - \frac{1}{4}\mathcal{P}^{aI}_\mu\mathcal{P}^\mu_{aI} \\
& +\frac{1}{4}e^{-2\phi}\left(\frac{1}{9}C^2 + C^I C^I + C^{aI}_1 C^{aI}_1 + C^{aI}_2 C^{aI}_2\right) \\
& -m^2 e^{-6\phi}\mathcal{M}_{00} + 2m^2 e^{-2\phi}\left(C\mathcal{V}_{00} - 3C^I\mathcal{V}_{0I}\right) \\
& -\frac{1}{64}\varepsilon^{\mu\upsilon\rho\sigma\lambda\tau}B_{\mu\upsilon}\left(\eta_{MN}\mathcal{F}^M_{\rho\sigma}\mathcal{F}^N_{\lambda\tau} + mB_{\rho\sigma}\mathcal{F}^0_{\lambda\tau} + \frac{1}{3}m^2 B_{\rho\sigma}B_{\lambda\tau}\right),
\end{aligned}
\tag{84}
$$

where m is an arbitrary mass parameter and

其中 m 是任意质量参数，且

$$
\mathcal{M}_{MN} = \mathcal{V}_M{}^m\mathcal{V}_N{}^m + \mathcal{V}_M{}^a\mathcal{V}_N{}^a,\ H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]},
$$

$$
\mathcal{F}^M_{\mu\upsilon} = 2\partial_{[\mu}A^M_{\upsilon]} + f_{NP}{}^M A^N_\mu A^P_\upsilon - m\delta^M_0 B_{\mu\upsilon} \tag{85}
$$

---

[11] Our $\mathcal{V}_M{}^A = \left(\mathcal{V}_M{}^0, \mathcal{V}_M{}^I, \mathcal{V}_M{}^a\right)$ and ψ^i_μ with $M, A = 1, \ldots, n+4, I = 1,2,3, a = 1, \ldots, n$ and $i = 1,2$ correspond to $L_\Lambda{}^\Sigma = \left(L_\Lambda{}^0, L_\Lambda{}^r, L_\Lambda{}^I\right)$ and ψ^A_μ in [108].

[11] 本文中我们带有 $M, A = 1, \ldots, n+4, I = 1,2,3, a = 1, \ldots, n$ 和 $i = 1,2$ 的 $\mathcal{V}_M{}^A = \left(\mathcal{V}_M{}^0, \mathcal{V}_M{}^I, \mathcal{V}_M{}^a\right)$ 与 ψ^i_μ 对应文献 [108] 中的 $L_\Lambda{}^\Sigma = \left(L_\Lambda{}^0, L_\Lambda{}^r, L_\Lambda{}^I\right)$ 和 ψ^A_μ 。

The supertransformations of the fermionic fields are given by [108]

费米子场的超变换由文献 [108] 给出:

$$
\begin{aligned}
\delta\psi^i_\mu = & D_\mu\varepsilon^i - \frac{1}{16}e^{-\varphi}\left[\mathcal{F}^M_{\rho\sigma}\mathcal{V}_M{}^0\varepsilon^{ij} - \mathcal{F}^M_{\rho\sigma}\mathcal{V}_M{}^I\left(\sigma^I\right)^{ij}\right]\left(\gamma_\mu{}^{\rho\sigma} - 6\delta^\rho_\mu\gamma^\sigma\right)\varepsilon_j \\
& +\frac{i}{32}e^{2\phi}H_{\rho\sigma\tau}\gamma_7\left(\gamma_\mu{}^{\rho\sigma\tau} - 3\delta^\rho_\mu\gamma^{\sigma\tau}\right)\varepsilon^i - S^{ij}\varepsilon_j \\
\delta\chi^i = & \frac{i}{2}\gamma^\mu\partial_\mu\phi\varepsilon^i + \frac{i}{16}\left[\mathcal{F}^M_{\mu\nu}\mathcal{V}_M{}^0\varepsilon^{ij} - \mathcal{F}^M_{\mu\nu}\mathcal{V}_M{}^I\left(\sigma^I\right)^{ij}\right]\gamma^{\mu\upsilon}\varepsilon_j \\
& -\frac{1}{32}e^{2\phi}\gamma^{\mu\upsilon\rho}H_{\mu\upsilon\rho}\gamma_7\varepsilon^i - N^{ij}\varepsilon_j
\end{aligned}
$$

$$\delta\lambda^{ai} = i\mathcal{P}_\mu^{aI}\left(\sigma^I\right)^{ij}\gamma^\mu\varepsilon_j + i\mathcal{P}^{a0}\varepsilon^{ij}\gamma^\mu\varepsilon_j + \frac{i}{2}\mathcal{F}_{\mu\nu}^M\mathcal{V}_M{}^a\varepsilon^i - M^{aij}\varepsilon_j, \tag{86}$$

where $D\varepsilon^i = D_\mu(\omega,\mathcal{Q})\,\varepsilon^i$ and the shift functions are

其中 $D\varepsilon^i = D_\mu(\omega,\mathcal{Q})\,\varepsilon^i$ ，位移函数为

$$S_{ij} = \frac{i}{24}e^\phi\left(C\varepsilon_{ij} - 3C^I\left(\sigma^I\right)_{ij}\right) + \frac{i}{4}me^{-3\phi}\left(\mathcal{V}_{00}\varepsilon_{ij} + \mathcal{V}_{I0}\left(\sigma^I\right)_{ij}\gamma_7\right),$$

$$N_{ij} = \frac{1}{24}e^\phi\left(C\varepsilon_{ij} + 3C^I\left(\sigma^I\right)_{ij}\right) - \frac{3}{4}me^{-3\phi}\left(\mathcal{V}_{00}\varepsilon_{ij} - \mathcal{V}_{I0}\left(\sigma^I\right)_{ij}\gamma_7\right),$$

$$M_{ij}^a = e^\phi\left(C_1^{aI} + 2i\gamma_7C_2^{aI}\right)\left(\sigma^I\right)_{ij} - 2me^{-2\phi}\mathcal{V}^a{}_0\varepsilon_{ij}. \tag{87}$$

(2, 0) Supergravity Coupled to Tensor Multiplets in 6D

(2, 0) 超引力与 6 维张量多重态耦合

The $(2,0)$, $6D$ supergravity multiplet has the field content $\{e_\mu^r, \psi_{\mu+}^i, 5B_{\mu\nu+}\}$ with the symplectic Majorana-Weyl gravitini in the 4-plet, and the two-form potential with anti-self-dual field strength in the 5-plet of the R-symmetry group $USp(4)_R$. Combining this with n copies of the tensor multiplets, each containing the fields $\{B_{\mu\nu-}, \chi_-, \phi^{ab}\}$, where the scalars in the 5-plet of $USp(4)_R$, we get the multiplet

$(2,0)$, $6D$ 超引力多重态的场内容为 $\{e_\mu^r, \psi_{\mu+}^i, 5B_{\mu\nu+}\}$ ，其中辛马约拉纳-外尔引力微子属于 R 对称性群 $USp(4)_R$ 的 4 维表示，带反自对偶场强的二形式势属于 R 对称性群 $USp(4)_R$ 的 5 维表示。将其与 n 份张量多重态结合，每份张量多重态包含场 $\{B_{\mu\nu-}, \chi_-, \phi^{ab}\}$ ，其中标量属于 $USp(4)_R$ 的 5 维表示，我们得到该多重态

$$\{e_\mu{}^r, B_{\mu\nu}^I, \mathcal{V}_M{}^A; \psi_{\mu+}^i, \chi_-^{ai}\}, \tag{88}$$

where $I = 1,\dots,n+5$ labels the fundamental of $SO(n,5)$, $a = 1,\dots,n$, $i = 1,\dots,4$ and $L_I{}^A$ is a representative of the $SO(n,5)/SO(n)\times SO(5)$ coset. This theory was constructed in [109, 110] by means of Noether procedure, and in [111] by using the superconformal tensor calculus. The coset representatives obey the defining relations

其中 $I = 1,\dots,n+5$ 标记 $SO(n,5)$, $a = 1,\dots,n$ 的基础表示，$i = 1,\dots,4$ 和 $L_I{}^A$ 是 $SO(n,5)/SO(n)\times SO(5)$ 陪集的一个代表。该理论由 [109, 110] 通过诺特定理过程构造，文献 [111] 则通过超共形张量微积分完成了构造。陪集代表满足以下定义关系

$$\mathcal{V}_M{}^a\mathcal{V}^M{}_{ij} = 0,\ \mathcal{V}_M{}^a\mathcal{V}^M{}_b = \delta_b^a,\ \mathcal{V}_M{}^{ij}\mathcal{V}^M{}_{k\ell} = -\delta_{[k}^{[i}\delta_{\ell]}^{j]} + \frac{1}{4}\Omega^{ij}\Omega_{k\ell}. \tag{89}$$

In terms of these elements, the scalar current, the composite connections, and the metric can be written as

利用这些元素，标量流、复合联络和度规可写为

$$P_\mu^{aij} = \mathcal{V}^{Ia}\partial_\mu \mathcal{V}_I^{ij},\ Q_\mu^{ab} = \mathcal{V}^{Ma}\partial_\mu \mathcal{V}_M{}^b,\ Q_{\mu i}{}^j = 2\mathcal{V}^M{}_{ik}\partial_\mu \mathcal{V}_M{}^{jk},$$

$$\mathcal{M}_{MN} = \mathcal{V}_M^{ij}\mathcal{V}_{Nij} + \mathcal{V}_M{}^a\mathcal{V}_{Na} \tag{90}$$

The $USp(4)$ indices are raised and lowered with the symplectic metric, and the indices M, N by the $SO(n,5)$ invariant metric $\eta_{MN} = -\mathcal{V}_M^{ij}\mathcal{V}_{Nij} + \mathcal{V}_M{}^a\mathcal{V}_{Na} = \mathrm{diag}(-1,-1,-1,-1,-1,+1,\ldots,+1)$. The bosonic part of the pseudo-Lagrangian is simply

$USp(4)$ 指标通过辛度量升降，指标 M, N 通过 $SO(n,5)$ 不变度量 $\eta_{MN} = -\mathcal{V}_M^{ij}\mathcal{V}_{Nij} + \mathcal{V}_M{}^a\mathcal{V}_{Na} = \mathrm{diag}(-1,-1,-1,-1,-1,+1,\ldots,+1)$ 升降。赝拉格朗日量的玻色子部分可简单表示为

$$e^{-1}\mathcal{L} = -\frac{1}{4}R + \frac{1}{12}\mathcal{M}_{IJ}H^I_{\mu\upsilon\rho}H^{\mu\upsilon\rho J} + P_\mu^{aij}P^\mu_{ij}, \tag{91}$$

where $H^I_{\mu\upsilon\rho} = 3\partial_{[\mu}B^I_{\upsilon\rho]}$. The supersymmetry transformations of the fermions are

其中 $H^I_{\mu\upsilon\rho} = 3\partial_{[\mu}B^I_{\upsilon\rho]}$。费米子的超对称变换为

$$\delta\psi_\mu^i = D_\mu\varepsilon^i - \frac{1}{12}H^{ij-}_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\gamma_\mu\varepsilon_j,\ \delta\chi^{ai} = \frac{1}{48}H^{+a}_{\mu\nu\rho}\gamma^{\mu\nu\rho}\varepsilon^i + \frac{1}{4}P_\mu^{aij}\varepsilon_j,$$

(92)

where $H^{ij} := H^M\mathcal{V}_M^{ij}, H^a := H^M\mathcal{V}_M^a$ and $D_\mu\varepsilon^i = \nabla_\mu\varepsilon^i + Q_\mu^{ij}\varepsilon_j$. We recall that the correct equations of motion are obtained by imposing the duality equations $H^{ij-}_{\mu\nu\rho} = 0$ and $H^{a+}_{\mu\nu\rho} = 0$ by hand, after varying the action with respect to all fields. The $(2,0)$ supergravity coupled to 21 tensor multiplet follows from Type IIB supergravity on $K3$, and it is anomaly free [112].

其中 $H^{ij} := H^M\mathcal{V}_M^{ij}, H^a := H^M\mathcal{V}_M^a$ 和 $D_\mu\varepsilon^i = \nabla_\mu\varepsilon^i + Q_\mu^{ij}\varepsilon_j$。我们记得，在对所有场变分作用量后，手动引入对偶条件 $H^{ij-}_{\mu\nu\rho} = 0$ 和 $H^{a+}_{\mu\nu\rho} = 0$ 即可得到正确的运动方程。耦合 21 个张量多重态的 $(2,0)$ 超引力可从 IIB 型超引力在 $K3$ 上约化得到，且无反常 [112]。

(1, 0) Supergravity Coupled to Vector, Tensor, and Hyper Multiplets in 6D

六维下耦合矢量、张量和超多重态的 (1, 0) 超引力

Generalities

概述

Starting with $N = (1,0), 6D$ supersymmetry, we begin to see the appearance of off-shell supergravity and matter multiplets, thanks to having only 8 fermionic generators. In particular, there are two versions the off-shell $(1,0), 6D$ supergravity. One of them is obtained from the coupling of the standard Weyl multiplet to a linear multiplet and fixing the dilatations, conformal boost and special supersymmetry transformations, and fixing a gauge that breaks $Sp(1)_R$ down to $U(1)_R$. The resulting off-shell multiplet containing $48_B + 48_F$ degrees of freedom is described by the fields [113]

从 $N = (1,0), 6D$ 超对称出发，由于仅存在 8 个费米子生成元，我们开始看到脱壳超引力与物质多重态的出现，具体而言，脱壳 $(1,0), 6D$ 超引力存在两种形式。其中一种是通过将标准外尔多重态与线性多重态耦合，固定伸缩变换、共形升压与特殊超对称变换，并选取一个将 $Sp(1)_R$ 破缺为 $U(1)_R$ 的规范得到的。最终得到的含 $48_B + 48_F$ 自由度的脱壳多重态由如下场描述 [113]

$$e_\mu{}^a\,(15)\,,\mathcal{V}'^{\,ij}_\mu\,(12)\,,\mathcal{V}_\mu\,(5)\,,B_{\mu\nu}\,(10)\,,\sigma\,(1)\,,E_{\mu\nu\rho\sigma}\,(5)\,;\psi_\mu{}^i\,(40)\,,\psi^i\,(8)\ ,$$

(93)

where $i = 1,2$, the vector tr $\mathcal{V}'^{ij}_\mu$ is symmetric and traceless, E is a 4-form potential and $\mathcal{V}_\mu$ is the gauge field of the surviving $U(1)_R$ gauge symmetry. The fermions (the last two in the list) are symplectic Majorana-Weyl. An alternative off-shell multiplet is obtained by coupling the dilaton Weyl multiplet to a linear multiplet and fixing the symmetries mentioned above in a slightly different way. This yields an off-shell multiplet in which σ is replaced by the trace of the linear multiplet scalars, $\delta^{ij}L_{ij}$, thus resulting again with 48+48 degrees of freedom. The 6D off-shell $\mathcal{N} = (1,0)$ supergravity was constructed in [113,114]. The off-shell formulation is very useful in constructing the higher derivative superinvariants. As our focus is on two-derivative supergravities here, we will set the auxiliary field to zero, by using their algebraic equations of motion, and review the resulting off-shell supergravities and their matter couplings.

其中对 $i = 1,2$ 而言，矢量 tr $\mathcal{V}'^{ij}_\mu$ 是对称无迹的，E 是 4-形式势，$\mathcal{V}_\mu$ 是剩余 $U(1)_R$ 规范对称性的规范场。费米子 (列表中最后两个) 是辛马约拉纳-外尔费米子。另一种脱壳多重态通过将伸缩子外尔多重态与线性多重态耦合，并以略有不同的方式固定上述对称性得到。由此得到的脱壳多重态中，σ 被替换为线性多重态标量的迹 $\delta^{ij}L_{ij}$ ，最终仍得到 $48+48$ 个自由度。6 维脱壳 $\mathcal{N} = (1,0)$ 超引力最早构造于文献 [113,114]。脱壳表述在构造高阶导数超不变量时非常有用。由于本文我们聚焦于二阶导数超引力，我们将利用辅助场的代数运动方程将其置零，然后综述由此得到的脱壳超引力及其物质耦合。

On-shell the $(1,0)$ super-Poincaré algebra in $6D$ admits the following multi-plets12

壳层上，$6D$ 维的 $(1,0)$ 超庞加莱代数存在如下多重态 [12]

$$\underbrace{(e^m_\mu, \psi^A_{\mu+}, B^-_{\mu\nu})}_{\text{graviton}},\ \underbrace{(B^+_{\mu\nu}, \chi^A_-, \varphi)}_{\text{tensor}},\ \underbrace{(A_\mu, \lambda^i_+)}_{\text{vector}},\ \underbrace{(4\phi, \psi_-)}_{\text{hyper}}. \tag{94}$$

The two-form potentials, $B^\pm_{\mu\nu}$, have (anti)self-dual field strengths. The spinors are symplectic Majorana-Weyl, $A = 1,2$ labels the doublet of the R symmetry group $Sp(1)_R$, and chiralities of the fermions are denoted by $\pm$.

二形式势 $B^{\pm}_{\mu\nu}$ 具有 (反) 自对偶场强。旋量是辛马约拉纳-外尔旋量，$A=1,2$ 标记 R 对称性群 $Sp(1)_R$ 的二重态，费米子的手征性由 ± 标记。

Hypersector and Quaternionic Kähler Manifolds

超 sector 与四元数凯勒流形

The couplings of two-derivative $(1,0)$ supergravity in $6D$ to a single tensor multiplet, n_V vector multiplets and n_H hypermultiplets, were given completely in [116]. As was first shown in [117], the locally supersymmetric coupling of hypermultiplets to supergravity requires that hyperscalars parametrize a quaternionic Kahler (QK) manifolds of negative scalar curvature. Such manifolds are typically noncompact [117]. All such spaces are necessarily Einstein. Those which are symmetric are exhausted by noncompact Wolf spaces. These spaces, listed in Table 6 under $3D, N=4$ supergravities, are

$6D$ 维下二阶导数 $(1,0)$ 超引力与单个张量多重态、n_V 个矢量多重态和 n_H 个超多重态的完整耦合已在文献 [116] 中给出。正如文献 [117] 首次证明的，超多重态与超引力的局域超对称耦合要求超标量参数化负标量曲率的四元数凯勒 (QK) 流形。这类流形通常是非紧的 [117]，且所有这类空间必然是爱因斯坦空间。对称的这类空间全部由非紧沃尔夫空间穷尽，这些空间已在表 6 的 $3D, N=4$ 超引力条目下列出，分别是

$$\frac{Sp(n,1)}{Sp(1)\times Sp(1)}, \frac{SU(n,2)}{SU(n)\times SU(2)\times U(1)}, \frac{SO(n,4)}{SO(n)\times SO(4)}, \frac{E_{8(-24)}}{E_7\times Sp(1)},$$
$$\frac{E_{7(-5)}}{SO(12)\times Sp(1)}, \frac{E_{6(2)}}{SU(6)\times Sp(1)}, \frac{F_{4(4)}}{Sp(3)\times Sp(1)}, \frac{G_{2(2)}}{SO(4)}. \tag{95}$$

---

[12] There also exists a linear multiplet which has a triplet of scalars L_{ij} , and a 4-form potential which is on-shell dual to a scalar field. In addition, there is a non-linear multiplet similar to the linear multiplet but in which the 3 scalars form an element of $SU(2)$. In both of these multiplets, the fermions are doublets of the R -symmetry group. See [113] for a discussion of the off-shell versions of these multiplets, and [113, 115] for some of their couplings.

[12] 此外还存在线性多重场，它包含一个标量三重态 L_{ij} ，以及一个壳层面上与标量场对偶的 4 形式势。另外还存在与线性多重场类似的非线性多重场，但其中 3 个标量构成 $SU(2)$ 的一个元素。这两类多重场中，费米子都是 R 对称群的二重态。关于这些多重场外形式的讨论见文献 [113]，部分耦合性质见文献 [113, 115]。

The comments in Table 6 are about how they are related to particular scalar manifold geometries in matter coupled $4D, N=2$ supergravity, known as (very) special Kähler, and those geometries that arise in $5D, N=2$ matter coupled supergravities, known as very special geometries; see for example, [118]. We shall comment further on these geometries where these $4D$ and $5D$ models are reviewed.

表 6 中的注释说明了它们与物质耦合 $4D, N = 2$ 超引力中特定标量流形几何 (即 (非常) 特殊凯勒几何) 的关系，也说明了 $5D, N = 2$ 物质耦合超引力中出现的几何 (即非常特殊几何)；相关例子可参见文献 [118]。我们在回顾这些 $4D$ 和 $5D$ 模型时会进一步讨论这些几何。

There also exists homogeneous but non-symmetric Alekseevsky spaces [119] whose classification was completed in [120]. [13] These are coset spaces G/H where G is a parabolic group and H is its maximal compact subgroup. The Lie algebra $\mathfrak{g}$ of the group G is a semi-direct sum

此外还存在齐次但非对称的阿列克谢耶夫斯基空间 [119]，其分类在文献 [120] 中完成。[13] 这类空间是陪集空间 G/H ，其中 G 是抛物群，H 是其极大紧子群。群 G 的李代数 $\mathfrak{g}$ 是半直和

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

$$\mathfrak{g}_0 = \mathrm{so}(1,1) \oplus \mathrm{so}(q+3,3) \oplus \mathcal{S}_q(P,Q),$$

$$\mathfrak{g}_1 = \big(SO(q+3,3) \text{ spinor}, \mathcal{S}_q(P,Q) \text{ vector }\big)_1,$$

$$\mathfrak{g}_2 = \big(SO(q+3,3) \text{ vector}, \mathcal{S}_q(P,Q) \text{ singlet }\big)_2, \tag{96}$$

where q, P, Q are positive integers or zero, the subscripts denote the $so(1,1)$ weights, and the group $\mathcal{S}_q(P,Q)$ is explained and given in Table 1 below [120]. The isotropy group H is

其中 q, P, Q 为正整数或零，下标表示 $so(1,1)$ 权，群 $\mathcal{S}_q(P,Q)$ 的说明见下表 1[120]。迷向群 H 为

$$H = SO(q+3) \otimes SU(2) \otimes \mathcal{S}_q(P,Q). \tag{97}$$

For the symmetric manifolds, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ are included. For example, taking $q = 8, P = Q = 0$, the generator count for $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2}$ is $14 + 64_c + (1+91) + 64_s + 14$ giving a total of 248 generators of the isometry group of the symmetric space $E_{8(-24)}/(E_7 \times Sp(1))$. This is a very special QK manifold. For more details, see [120].

对于对称流形，包含 $\mathfrak{g}_{-1}$ 和 $\mathfrak{g}_{-2}$ 。例如，取 $q = 8, P = Q = 0$ 时，$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2}$ 的生成元计数为 $14 + 64_c + (1+91) + 64_s + 14$ ，得到对称空间 $E_{8(-24)}/(E_7 \times Sp(1))$ 等距群的生成元总数为 248。这是一个非常特殊的 QK 流形。更多细节参见文献 [120]。

The QK manifolds have tangent space group $Sp(n_H) \times Sp(1)_R$, and one can introduce vielbeins V_X^{ri} and their inverse $V_{ri}^{\bar{X}}$ satisfying [14]

QK 流形的切空间群为 $Sp(n_H) \times Sp(1)_R$ ，可以引入标架 V_X^{ri} 及其逆 $V_{ri}^{\bar{X}}$ ，满足 [14]

$$g_{\alpha\beta} V_{aA}^{\alpha} V_{bB}^{\beta} = \Omega_{ab}\varepsilon_{AB}, \; V_{aA}^{\alpha} V^{\beta bB} + (\alpha \leftrightarrow \beta) = g^{\alpha\beta}\delta_A^B, \tag{98}$$

where $g_{\alpha\beta}$ is the metric. According to [117] it is not known if there are negatively curved QK manifolds which do not admit globally defined vielbeins V_a^{aA} . An $Sp(n_H) \times Sp(1)_R$ valued connection is defined through the vanishing torsion condition

其中 $g_{\alpha\beta}$ 为度规。根据文献 [117]，目前尚不清楚是否存在不允许整体定义标架 V_a^{aA} 的负曲率四元数凯勒流形。通过挠率为零条件可定义一个取值在 $Sp(n_H) \times Sp(1)_R$ 的联络

$$\partial_\alpha V_{\beta aA} + A_{\alpha a}{}^b V_{\beta bA} + A_{\alpha A}{}^B V_{\beta aB} - (\alpha \leftrightarrow \beta) = 0. \tag{99}$$

---

[13] For non-homogeneous QK manifolds, see [121,122].

[13] 非齐次四元数凯勒流形参见文献 [121,122]。

[14] Note that the third relation in eq. (2) of [117], which also appeared in several papers that followed, holds only for $n = 1$, namely the case of a single hypermultiplet.

[14] 需要注意，文献 [117] 式 (2) 中的第三个关系 (该关系也出现在后续多篇论文中) 仅对 $n = 1$ 成立，也就是单个超多重子的情况。

Table 1 Real Clifford algebras $\mathcal{C}(q+1,0)$. Here $\mathbb{F}(n)$ stands for $n \times n$ matrices with entries over the field $\mathbb{F}$, and $\mathcal{D}_{q+1}$ denotes the real dimensions of an irreducible representation of the Clifford algebra. $\mathcal{S}_q(P,Q)$ is the metric preserving group in the centralizer of the Clifford algebra in the $(P+Q)\,\mathcal{D}_{q+1}$ dimensional representation

表 1 实克利福德代数 $\mathcal{C}(q+1,0)$ 。其中 $\mathbb{F}(n)$ 表示元素取自域 $\mathbb{F}$ 的 $n \times n$ 矩阵，$\mathcal{D}_{q+1}$ 表示克利福德代数不可约表示的实维数，$\mathcal{S}_q(P,Q)$ 是 $(P+Q)\,\mathcal{D}_{q+1}$ 维表示中克利福德代数中心化子里的保度规群

q	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathcal{S}_q(P,Q)$
-1	R	1	$SO(P)$
0	$\mathbb{R} \oplus \mathbb{R}$	1	$SO(P) \otimes SO(Q)$
1	$\mathbb{R}(2)$	2	$SO(P)$
2	$\mathbb{C}(2)$	4	$U(P)$
3	$\mathbb{H}(2)$	8	$USp(2P)$
4	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	8	$USp(2P) \otimes USp(2Q)$
5	$\mathbb{H}(4)$	16	$USp(2P)$
6	$\mathbb{C}(8)$	16	$U(P)$
7	$\mathbb{R}(16)$	16	$SO(P)$
$n+8$	$\mathbb{R}(16) \otimes \mathcal{C}(n+1,0)$	16 $\mathcal{D}_n$	as for $q = n$

From the fact that the vielbein V_{aA}^α is covariantly constant, one derives that

由标架 V_{aA}^α 协变常数这一性质，可以推导出

$$R_{\alpha\beta\gamma\delta}V_{aA}^{\delta}V_{bB}^{\gamma}=\varepsilon_{AB}F_{\alpha\beta ab}+\Omega_{ab}F_{\alpha\beta aB}, \tag{100}$$

where F_{AB} and F_{ab} are the curvature two-forms of $Sp(1)_R$ and $Sp\left(n_H\right)$ connection, respectively. The manifold has a quaternionic Kähler structure characterized by three locally defined $(1,1)$ tensors $J^r{}_\alpha{}^\beta$ $(r=1,2,3)$ satisfying the quaternion algebra

> 其中 F_{AB} 和 F_{ab} 分别是 $Sp(1)_R$ 联络和 $Sp\left(n_H\right)$ 联络的曲率二形式。该流形具有四元数凯勒结构，其特征是满足四元数代数的三个局部定义的 $(1,1)$ 张量 $J^r{}_\alpha{}^\beta$ $(r=1,2,3)$

$$J^r{}_\alpha{}^\beta J^s{}_\beta{}^\gamma=-\delta^{rs}\delta_\alpha^\gamma+\varepsilon^{rst}J^t{}_\alpha{}^\gamma. \tag{101}$$

These tensors can be expressed as $J^r{}_\alpha{}^\beta=-i(\sigma^r)_A{}^B V_\alpha^{aA}V_{aB}^\beta$, where σ^r are the Pauli matrices. We can define a triplet of two-forms $J_{\alpha\beta}^r=J^r{}_\alpha{}^\gamma g_{\gamma\beta}$, and these are covariantly constant, namely, $\nabla_\alpha J_{\beta\gamma}^r+\varepsilon^{rst}A_\alpha^s J_{\beta\gamma}^t=0$, with $A_\alpha^r\equiv\frac{i}{2}(\sigma^r)_A{}^B A_{\alpha B}{}^A$. For $n_H>1$, quaternionic Kähler manifolds are Einstein spaces, i.e., $R_{\alpha\beta}=\lambda g_{\alpha\beta}$. Using the covariant constancy of $J_{\alpha\beta}^r$, one finds that [123]

> 这些张量可表示为 $J^r{}_\alpha{}^\beta=-i(\sigma^r)_A{}^B V_\alpha^{aA}V_{aB}^\beta$ ，其中 σ^r 是泡利矩阵。我们可以定义一组三元二形式 $J_{\alpha\beta}^r=J^r{}_\alpha{}^\gamma g_{\gamma\beta}$ ，它们是协变常数的，即 $\nabla_\alpha J_{\beta\gamma}^r+\varepsilon^{rst}A_\alpha^s J_{\beta\gamma}^t=0$ ，满足 $A_\alpha^r\equiv\frac{i}{2}(\sigma^r)_A{}^B A_{\alpha B}{}^A$ 。当 $n_H>1$ 时，四元数凯勒流形是爱因斯坦空间，即 $R_{\alpha\beta}=\lambda g_{\alpha\beta}$ 。利用 $J_{\alpha\beta}^r$ 的协变常数性质，可以得到 [123]

$$F_{\alpha\beta}^r=\frac{\lambda}{n_H+1}J_{\alpha\beta}^r. \tag{102}$$

Local supersymmetry relates λ to the gravitational coupling constant (which we set to 1) and requires that $\lambda<0$ [117], explicitly $\lambda=-\left(n_H+1\right)$. For $n_H=1$ all Riemannian 4-manifolds are quaternionic Kähler. [15] Using this value, substitution of (102) into (100) and the use of the curvature cyclic identity gives [117]

> 局域超对称将 λ 与引力耦合常数 (我们将其取为 1) 联系起来，并要求满足 $\lambda<0$ [117]，具体形式为 $\lambda=-\left(n_H+1\right)$ 。当 $n_H=1$ 时，所有黎曼 4-流形都是四元数凯勒流形。[15] 代入该数值后，将 (102) 代入 (100)，再利用曲率循环恒等式可得 [117]

$$F_{\alpha\beta ab}=V_{[\alpha}^{cA}V_{\beta]A}^d\left(-2\Omega_{ca}\Omega_{db}+\Omega_{abcd}\right), \tag{103}$$

where Ω_{abcd} is a totally symmetric tensor defined by this equation. For symmetric QK manifolds, $\Omega_{abcd}=0$.

> 其中 Ω_{abcd} 是由该方程定义的全对称张量。对于对称四元数凯勒流形，有 $\Omega_{abcd}=0$ 。

The Case of Single Tensor Multiplet

> ## 单个张量多重态的情况

In the case of quaternionic projective space $Hp\,(n_H) = Sp\,(n_H, 1)\,/Sp\,(n_H)\times\ Sp(1)_R$, the maximal compact subgroup of its isometry group was gauged in [116], where couplings to Yang-Mills and hypermultiplets were completely determined. The full multiplet content in this case is given by

对于四元射影空间 $Hp\,(n_H) = Sp\,(n_H, 1)\,/Sp\,(n_H)\times\ Sp(1)_R$ ，文献 [116] 已对其等距群的极大紧致子群做了规范处理，并完全确定了它与杨-米尔斯场和超多重态的耦合。此情形下的完整多重态内容如下:

$$\left(e_\mu{}^m, \psi_\mu^A, B_{\mu\nu}, \chi^A, \varphi\right),\ \left(A_\mu^{\hat{I}}, \lambda^{\hat{I}A}\right),\ (\phi^\alpha, \psi^a),\ \hat{I} = (I, i),\ i = 1, 2, 3,$$

$$a = 1, \dots, n_H, \tag{104}$$

where I labels the adjoint representation of $Sp\,(n_T)$, and $\alpha = 1, \dots, 4n_H$ labels the coordinates of $Hp\,(n_H)$. Here we have combined the $(1, 0)$ supermultiplet with the single tensor multiplet, resulting in what we may think of as ”reducible supergravity multiplet.” Further building blocks needed are defined as follows [116],

其中 I 标记 $Sp\,(n_T)$ 的伴随表示，$\alpha = 1, \dots, 4n_H$ 标记 $Hp\,(n_H)$ 的坐标。我们在此将 $(1, 0)$ 超多重态与单个张量多重态结合，得到了可称之为“可约超引力多重态”的结构。所需的其他构造块定义如下 [116]:

$$\mathcal{P}_\mu^{aA} = D_\mu\phi^\alpha V_\alpha^{aA},\ \mathcal{Q}_\mu^{AB} = D_\mu\phi^\alpha A_\alpha^{AB} + A_\mu^{AB},\ \mathcal{Q}_\mu^{ab} = D_\mu\phi^\alpha A_\alpha^{ab} + A_\mu^{ab},$$

$$\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + 6v_z \mathrm{tr}_z\left(A_{[\mu}\partial_\nu A_{\rho]} + \frac{2}{3}A_{[\mu}A_\nu A_{\rho]}\right),$$

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - \mathrm{tr}\left(A_\mu K^\alpha\right)$$

$$C^i \equiv C^{i\hat{I}}T^{\hat{I}},\ C^{iI} = A_\alpha^i K^{\alpha I},\ C^{ij} = A_\alpha^i K^{\alpha j} - \delta^{ij}, \tag{105}$$

where $F = F^{\hat{I}}T^{\hat{I}}, \mathrm{tr}\left(T^{\hat{I}}T^{\hat{J}}\right) = \delta^{\hat{I}\hat{J}}$, the Killing vectors $K^\alpha = K^{\alpha\hat{I}}T^{\hat{I}}$, and $v_z = (v_1, v_2)$ with $v_1 = 1/g^2$ and $v_2 = 1/g'^2$ representing the coupling constants of $Sp\,(1)$ and $Sp(n_H)_R$, respectively. The bosonic part of the complete Lagrangian is given by [116][16]

其中 $F = F^{\hat{I}}T^{\hat{I}}, \mathrm{tr}\left(T^{\hat{I}}T^{\hat{J}}\right) = \delta^{\hat{I}\hat{J}}$ ，基林矢量 $K^\alpha = K^{\alpha\hat{I}}T^{\hat{I}}$ 和 $v_z = (v_1, v_2)$ ，而 $v_1 = 1/g^2$ 与 $v_2 = 1/g'^2$ 分别代表 $Sp\,(1)$ 和 $Sp(n_H)_R$ 的耦合常数。完整拉格朗日量的玻色子部分为 [116][16]

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{12}e^{2\varphi}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \frac{1}{4}e^\varphi v_z \mathrm{tr}_z\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{2}\mathcal{P}_\mu^{aA}\mathcal{P}_{aA}^\mu$$

$$-\frac{1}{8}e^{-\varphi}v_z^{-1}\mathrm{tr}_z\left(C^iC^j\right) \tag{106}$$

[15] Sometimes (102) is used to extend the definition of the quaternionic Kähler to $4D$, which restricts the manifold to be Einstein and self-dual [123].

[15] 有时 (102) 会被用来将四元数凯勒流形的定义推广到 $4D$ ，这会限制该流形为爱因斯坦自对偶流形 [123]。

[16] We use the conventions of [116] with the replacements $\eta_{mn} \to -\eta_{mn}, \gamma^m \to i\gamma^m, \varphi \to \varphi/\sqrt{2}, \lambda^i \to \sqrt{2}\lambda^i/g, \lambda^I \to \sqrt{2}\lambda^I/g', A_\mu \to A_\mu/g$. Note also that while a coupling of the form $\Omega_{abcd}\overline{\psi}^a\psi^b\overline{\psi}^c\psi^d$ is given in [116], it is present only for non-symmetric QK manifolds.

[16] 我们采用文献 [116] 中的约定，并做替换 $\eta_{mn} \to -\eta_{mn}, \gamma^m \to i\gamma^m, \varphi \to \varphi/\sqrt{2}, \lambda^i \to \sqrt{2}\lambda^i/g, \lambda^I \to \sqrt{2}\lambda^I/g', A_\mu \to A_\mu/g$。另外需要注意，尽管文献 [116] 给出了 $\Omega_{abcd}\overline{\psi}^a\psi^b\overline{\psi}^c\psi^d$ 形式的耦合，该耦合仅存在于非对称四元数凯勒流形上。

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and the local supersymmetry transformations are given by

局部超对称变换为:

$$\delta\psi_\mu^A = D_\mu\varepsilon^A + \frac{1}{24}e^\varphi\mathcal{H}_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\gamma_\mu\varepsilon^A,$$

$$\delta\chi^A = \frac{1}{2}\gamma^\mu\varepsilon^A\partial_\mu\varphi - \frac{1}{12}e^\varphi\mathcal{H}_{\mu\nu\rho}\gamma^{\mu\nu\rho}\varepsilon^A,$$

$$\delta\lambda^{iA} = \frac{1}{4}e^{\varphi/2}F^i_{\mu\nu}\gamma^{\mu\nu}\varepsilon^A - \frac{1}{2}ge^{-\varphi/2}C^{iAB}\varepsilon_B,$$

$$\delta\lambda^{IA} = \frac{1}{4}e^{\varphi/2}F^I_{\mu\nu}\gamma^{\mu\nu}\varepsilon^A - \frac{1}{2}g'e^{-\varphi/2}C^{IAB}\varepsilon_B,$$

$$\delta\psi^a = -\mathcal{P}^{aA}_\mu\gamma^\mu\varepsilon_A \tag{107}$$

where $D_\mu\varepsilon^A = \nabla_\mu\varepsilon^A + \mathcal{Q}^{AB}_\mu\varepsilon_B, C^{iAB} = C^{ij}\left(T^j\right)^{AB}$ and $C^{IAB} = C^{iI}\left(T^i\right)^{AB}$. Passing to a "string frame" entails the following steps [124]

其中 $D_\mu\varepsilon^A = \nabla_\mu\varepsilon^A + \mathcal{Q}^{AB}_\mu\varepsilon_B, C^{iAB} = C^{ij}\left(T^j\right)^{AB}$ 且 $C^{IAB} = C^{iI}\left(T^i\right)^{AB}$ 。转到“弦框架”需要以下步骤 [124]

$$e_\mu{}^m \to e^{\varphi/2}e_\mu{}^m, \psi_\mu \to e^{\varphi/4}\left(\psi_\mu + \frac{1}{2}\gamma_\mu\chi\right), (\chi, \lambda, \psi^a) \to e^{-\varphi/4}(\chi, \lambda, \psi^a),$$

$$\varepsilon \to e^{\varphi/4}\varepsilon, \delta_\varepsilon + \delta_\Lambda \to \delta_\lambda,\ \lambda^m{}_n = \frac{1}{2}\bar{\varepsilon}\gamma^m{}_n\chi, \tag{108}$$

where δ_ε and δ_Λ are the supersymmetry and local Lorentz transformations, and gives

其中 δ_ε 和 δ_Λ 分别是超对称变换和局部洛伦兹变换，由此可得

$$\mathcal{L} = ee^{2\varphi}\Big[\frac{1}{4}R - \frac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{12}H_{\mu\upsilon\rho}H^{\mu\upsilon\rho} - \frac{1}{4}\upsilon_z \mathrm{tr}_z\left(F_{\mu\upsilon}F^{\mu\upsilon}\right) - \frac{1}{2}P^{aA}_\mu P^\mu_{aA}$$

$$-\frac{1}{8}\upsilon_z^{-1}\mathrm{tr}_z\left(C^iC^j\right)\Big] \tag{109}$$

with the supertransformations of the fermionic fields given by

费米子场的超变换为

$$\delta\psi^A_\mu = D_\mu\left(\omega_-\right)\varepsilon^A$$

$$\delta\chi^A = \frac{1}{2}\gamma^\mu\varepsilon^A\partial_\mu\varphi - \frac{1}{12}H_{\mu\upsilon\rho}\gamma^{\mu\upsilon\rho}\varepsilon^A,$$

$$\delta\lambda^{iA} = \frac{1}{4}F^i_{\mu\nu}\gamma^{\mu\upsilon}\varepsilon^A - \frac{1}{2}gC^{iAB}\varepsilon_B$$

$$\delta\lambda^{IA} = \frac{1}{4}F^I_{\mu\nu}\gamma^{\mu\nu}\varepsilon^A - \frac{1}{2}g'C^{IAB}\varepsilon_B,$$

$$\delta\psi^a = -P^{aA}_\mu\gamma^\mu\varepsilon_A \tag{110}$$

where $C^{\hat{I}AB} = C^{i\hat{I}}\left(T^i\right)^{AB}$ and the field strength H has been absorbed into the definition of the spin connection as torsion so that $\omega_{-\mu ab} := \omega_{\mu ab} - H_{\mu ab}$.

其中 $C^{\hat{I}AB} = C^{i\hat{I}}\left(T^i\right)^{AB}$ 和场强 H 已作为挠率吸收进自旋联络的定义中，因此得到 $\omega_{-\mu ab} := \omega_{\mu ab} - H_{\mu ab}$。

$U(1)_R$ Gauged Einstein-Maxwell Supergravity

$U(1)_R$ 规范爱因斯坦-麦克斯韦超引力

The special case in which the hypermultiplets are left out and only a single vector multiplet is kept to gauge $U(1)_R \subset Sp(1)_R$ [125] has attracted much interest, finding applications in cosmology and phenomenology [126-128]. In this case the bosonic part of the Lagrangian, in the conventions of [125], is given by

去掉超多重态，仅保留单个矢量多重态来规范 $U(1)_R \subset Sp(1)_R$ 的特殊情况 [125] 引发了大量研究兴趣，它已被应用于宇宙学和唯象学中 [126-128]。在此情况下，采用文献 [125] 的约定，拉格朗日量的玻色子部分为

$$e^{-1}\mathcal{L} = \frac{1}{4\kappa^2}R - \frac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{12}e^{2\kappa\varphi}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \frac{1}{4}e^{\kappa\varphi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g^2\kappa^{-4}e^{-\kappa\varphi},$$

(111)

and the supertransformations of the fermions are

费米子的超变换为

$$\delta\psi_\mu = \kappa^{-1} D_\mu \varepsilon + \frac{1}{24} e^{\kappa\varphi} \mathcal{H}_{\upsilon\rho\sigma} \gamma^{\upsilon\rho\sigma} \gamma_\mu \varepsilon$$

$$\delta\chi = -\frac{1}{2} \partial_\mu \varphi \gamma^\mu \varepsilon + \frac{1}{12} e^{\kappa\varphi} \mathcal{H}_{\mu\upsilon\rho} \gamma^{\mu\upsilon\rho} \varepsilon$$

$$\delta\lambda = \frac{1}{2\sqrt{2}} e^{\kappa\varphi/2} F_{\mu\upsilon} \gamma^{\mu\upsilon} \varepsilon - \frac{i}{\sqrt{2}} g e^{\kappa\varphi/2} \varepsilon, \tag{112}$$

where the fermions are Weyl, $F = dA$ and $\mathcal{H}_{\mu\upsilon\rho} = 3\partial_{[\mu} B_{\upsilon\rho]} + 3\kappa F_{[\mu\upsilon} A_{\rho]}$. The Minkowski $\times S^2$ compactification was found in [125] and dyonic string solution in [129].

其中费米子是外尔费米子，满足 $F = dA$ 和 $\mathcal{H}_{\mu\upsilon\rho} = 3\partial_{[\mu} B_{\upsilon\rho]} + 3\kappa F_{[\mu\upsilon} A_{\rho]}$。闵氏 $\times S^2$ 紧致化由文献 [125] 得到，弦 dynonic 解由文献 [129] 得到。

The $U(1)_R$ gauged $N = (1, 0)$, $6D$ supergravity by itself is anomalous. The simplest way to remedy this is to couple to 245 hypermultiplets. Anomaly freedom by more general couplings to a suitable set of vector, hyper and tensor multiplets, turns out to be rare if one insists on gauge groups other than $U(1)$ and $SU(2)$. It is still not known if the few such models obtained so far [130- 133], can embedded in string/M theory. As a progress towards this end, it was shown in [134] that a dimensional reduction of the (anomalous) pure $N = 1$, 10D supergravity on a noncompact 3-manifold $H_{2,2}$ which can be embedded in a plane with $(2, 2)$ signature and which has $U(1) \times U(1)$ isometry, yields an $SO(2, 2)$ gauged supergravity in $7D$, whose reduction on a circle followed by a chiral truncation yields the $U(1)_R$ gauged supergravity in $6D$. It has been shown that the $SO(2, 1)$ and $SO(3, 1)$ gauged half-maximal supergravities in $7D$, reviewed in section "Half-Maximal 7D Gauged Supergravity Coupled to Vector Multiplets", also admit circle reduction followed by chiral truncation that yield the $U(1)_R$ gauged supergravity in $6D$ [100]. It has also been shown that the $N = (1, 1)$, $6D$ model obtained from a generalized dimensional reduction of pure half-maximal $7D$ supergravity, and whose equations of motion cannot be derived from a Lagrangian, has a truncation, albeit nonsupersymmetric one, that yields the equations of motion of the $U(1)_R$ gauge Maxwell-Einstein theory [106].

经 $U(1)_R$ 规范的 $N = (1, 0)$, $6D$ 超引力本身是反常的。解决该问题最简单的方法是耦合 245 个超多重态。如果要求规范群不为 $U(1)$ 和 $SU(2)$, 那么通过更一般的耦合到一组合适的矢量、超和张量多重态来消除反常的情况其实非常少。目前尚不清楚迄今得到的少数这类模型 [130-133] 能否嵌入弦/M 理论。作为该方向的进展，文献 [134] 证明:(反常) 纯 $N = 1$, 10D 超引力在非紧 3 流形 $H_{2,2}$ 上维约化后, 可得到 $SO(2, 2)$ 规范超引力, 其中该 3 流形可以嵌入带 $(2, 2)$ 号差的平面, 且具有 $U(1) \times U(1)$ 等距性；再将该超引力圆约化后做手征截断，就得到了 $6D$ 维的 $U(1)_R$ 规范超引力。已有研究证明, " 半最大 7 维耦合矢量多重态规范超引力" 一节中回顾的 $7D$ 维 $SO(2, 1)$ 和 $SO(3, 1)$ 规范半最大超引力，同样可以先做圆约化再做手征截断，得到 $6D$ 维的 $U(1)_R$ 规范超引力 [100]。还有研究证明, 从纯半最大 $7D$ 超引力广义维约化得到的 $N = (1, 1)$, $6D$ 模型 (其运动方程无法从拉格朗日量导出) 存在一个截断——尽管它不具有超对称性——可以给出 $U(1)_R$ 规范麦克斯韦-爱因斯坦理论的运动方程 [106]。

The Case of Multi-Tensor Multiplets

多张量多重态情况

Coupling an arbitrary number of tensor multiplets to $N = (1,0)$, $6D$ supergravity brings in a number of new features and subtleties. In that case, the scalar fields of the tensor multiplets parametrize the coset $SO(n_T,1)/SO(n_T)$, and the chiral two-form of supergravity multiplet together with the n_T anti-chiral two-forms coming from the tensor multiplets transform a vector of $SO(n_T+1)$. The field equations of multi-tensors coupled to $(1,0)$ supergravity in leading order in fermions were obtained in [109]. The vector fields were included in [135], and the hypermultiplet couplings as well in [136], where the complete supertransformations were also found. Subsequently, the complete field equations without hypermultiplets were found in [137]. Finally, the results of [136, 137] were completed to include the higher order fermion terms in [138], where a pseudo-Lagrangian was given as well.

将任意数量的张量多重态耦合到 $N = (1,0)$, $6D$ 超引力会带来许多新特性和细微问题。这种情况下，张量多重态的标量场参数化陪集 $SO(n_T,1)/SO(n_T)$， 超引力多重态的手征二形式与来自张量多重态的 n_T 反手征二形式共同构成 $SO(n_T+1)$ 的矢量变换。多张量耦合 $(1,0)$ 超引力的场方程是在费米子领头阶下由文献 [109] 得到的；矢量场由文献 [135] 纳入，超多重态耦合也在文献 [136] 中被加入，该文献同时还找到了完整的超变换。随后，不含超多重态的完整场方程在文献 [137] 中得到。最终，文献 [136, 137] 的结果在文献 [138] 中被补全，加入了高阶费米子项，该文献同时还给出了一个伪拉格朗日量。

In the rest of this subsection, we leave out the hypermultiplets and combine n_T copies of the tensor multiplet consisting of the fields $(B_{\mu\nu}, \varphi, \lambda)$ with the pure supergavity multiplet, and consider Yang-Mills sector with gauge a semi-simple gauge group $G = \Pi_z G_z$, without R-symmetry gauging. For the field content, we introduce the notation

在本小节余下部分，我们忽略超多重态，将由场 $(B_{\mu\nu}, \varphi, \lambda)$ 构成的 n_T 份张量多重态与纯超引力多重态结合，讨论规范群为半单规范群 $G = \Pi_z G_z$ 的杨-米尔斯 sector，不进行 R 对称性定域规范。对于场内容，我们引入记号

$$\left\{e_\mu{}^m, B^I_{\mu\nu}, L_I{}^A; \psi^i_\mu, \chi^r\right\}, \left\{A_\mu, \lambda\right\}_z, \tag{113}$$

where $I, A = 0, 1, \ldots, n_T, r = 1, \ldots, n_T$, and $L^A_M = \left(L_I{}^0, L_I{}^r\right)$ is the $SO(n_T,1)/SO(n_T)$ coset representative parametrized by n_T tensor multiplet scalars. The fermions are symplectic Majorana-Weyl. The $SO(n_T,1)$ invariant tensor $\eta = \mathrm{diag}(-1,+1,\ldots,+1)$ and the positive definite scalar matrix $\mathcal{M}$ are given by

其中 $I, A = 0, 1, \ldots, n_T, r = 1, \ldots, n_T$， $L^A_M = \left(L_I{}^0, L_I{}^r\right)$ 是由 n_T 个张量多重态标量参数化的 $SO(n_T,1)/SO(n_T)$ 陪集代表元。费米子是辛马约拉纳-外尔费米子。$SO(n_T,1)$ 不变张量 $\eta = \mathrm{diag}(-1,+1,\ldots,+1)$ 和正定标量矩阵 $\mathcal{M}$ 由下式给出

$$\eta_{IJ} = -L_I{}^0 L_J{}^0 + L_I{}^r L_J{}^r, \; \mathcal{M}_{IJ} = L_I{}^0 L_J{}^0 + L_I{}^r L_J{}^r. \tag{114}$$

Suppressing the fermionic field dependence, the self-duality equation is

忽略费米场依赖，自对偶方程为

$$\mathcal{M}_{IJ}H^{J\mu\nu\rho}=\frac{1}{6}\eta_{IJ}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}H^{J}_{\sigma\lambda\tau}, \tag{115}$$

where

其中

$$H^{I}_{\mu\nu\rho}=3\partial_{[\mu}B_{\nu\rho]}+6c^{Iz}\mathrm{tr}_z\left(A_{[\mu}\partial_\nu A_{\rho]}+\frac{2}{3}A_{[\mu}A_\nu A_{\rho]}\right). \tag{116}$$

The Yang-Mills equation obtained from supersymmetry considerations, and suppressing the fermionic terms, is given by

从超对称性考虑得到的杨-米尔斯方程，忽略费米子项后写为

$$D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=\frac{1}{2}c^{Iz}\mathcal{M}_{IJ}H^{J\upsilon\rho\sigma}F^{z}_{\rho\sigma}, \tag{117}$$

where c^{Iz} are coupling constants and we have defined ${L_I}^0\equiv L_I$. Writing this equation as $D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=J_z^\upsilon$, and taking the covariant divergence of both sides, and using the duality equation, one finds that the current J_z^μ fails to be conserved [135], since [17]

其中 c^{Iz} 是耦合常数，我们已经定义了 ${L_I}^0\equiv L_I$ 。将该方程写为 $D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=J_z^\upsilon$ ，对两边取协变散度并利用对偶方程，可以发现流 J_z^μ 不满足守恒律 [135]，这是因为 [17]

$$D_\mu J_z^\mu=\frac{1}{16}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}\eta_{IJ}c^{Iz}c^{Jz'}F_{z\mu\nu}\mathrm{tr}_{z'}\left(F_{\rho\sigma}F_{\lambda\tau}\right). \tag{118}$$

In view of this (covariant) anomaly, the Yang-Mills equation of motion cannot be derived from an action, pseudo or not. The presence of the anomaly in the gauge symmetry also implies, by supersymmetry algebra that it must imply the presence of supersymmetry anomaly as well. A useful way to characterize both supersymmetry and gauge anomalies is to integrate all the field equations, except those of the Yang-Mills sector, into a pseudo action and add the Wess-Zumino term of the form $B\,(\wedge\,\mathrm{tr}\,F\wedge F)$ which will modify the Yang-Mills equation (117) such that it yields the so-called consistent anomaly. The gauge anomaly thus manifests itself as the non-vanishing of the gauge variation of the Wess-Zumino term, and the expected supersymmetry anomaly, as its non-vanishing variation under supersymmetry [137]. Moreover, these anomalies have been shown [137] to obey the Wess-Zumino consistency conditions

鉴于这个 (协变) 反常，杨-米尔斯运动方程无法从任何作用量 (无论是伪作用量还是普通作用量) 导出。规范对称性中存在反常也意味着，根据超对称代数，必然也存在超对称反常。表征超对称反常和规范反常的一种有效方法是，将杨-米尔斯 sector 以外所有场方程整合到一个伪作用量中，再加入形式为 $B\,(\wedge\,\mathrm{tr}\,F\wedge F)$ 的韦斯-祖米诺项，它会修改杨-米尔斯方程 (117)，使之得到所谓的相容反常。规范反常体现为韦斯-祖米诺项的规范变分非零，而预期的超对称反常体现为该项在超对称下的变分非零 [137]。此外文献 [137] 已经证明，这些反常满足韦斯-祖米诺相容性条件

$$\delta_{\Lambda_1}\mathcal{A}_{\Lambda_2} - \delta_{\Lambda_2}\mathcal{A}_{\Lambda_1} = \mathcal{A}_{\Lambda_3},\ \delta_{\Lambda}\mathcal{A}_{\varepsilon} = \delta_{\varepsilon}\mathcal{A}_{\Lambda}$$

$$\delta_{\varepsilon_1}\mathcal{A}_{\varepsilon_2} - \delta_{\varepsilon_2}\mathcal{A}_{\varepsilon_1} = \mathcal{A}_{\Lambda} + \mathcal{A}_{\varepsilon_3} \tag{119}$$

where $\mathcal{A}_{\Lambda}$ and $\mathcal{A}_{\varepsilon}$ denote the gauge and supersymmetry anomalies, respectively.

其中 $\mathcal{A}_{\Lambda}$ 和 $\mathcal{A}_{\varepsilon}$ 分别表示规范反常和超对称反常。

The bosonic part of the pseudo-Lagrangian is given by [137][18]

伪拉格朗日量的玻色部分由 [137][18] 给出

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{1}{48}\mathcal{M}_{IJ}H^{I\mu\upsilon\rho}H^{J}_{\mu\upsilon\rho} - \frac{1}{4}\partial_{\mu}L_{I}\partial^{\mu}L^{I} - \frac{1}{4}c^{I}_{z}L_{I}\mathrm{tr}_{z}\left(F_{\mu\upsilon}F^{\mu\upsilon}\right)$$

$$-\frac{1}{32}\varepsilon^{\mu\upsilon\rho\sigma\lambda\tau}\eta_{IJ}c^{Iz}B^{J}_{\mu\upsilon}\mathrm{tr}_{z}\left(F_{\rho\sigma}F_{\lambda\tau}\right). \tag{120}$$

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[17] In presence of hypermultiplets, there will be a contribution to the current coming from the hypermultiplet scalars which is conserved.

[17] 当存在超多重态时，守恒流会来自超多重态标量的贡献。

[18] Here we go from the conventions of [137] to those of [136] by letting $c^{I}_{z} \to -c^{I}_{z}/2, H_{\mu\upsilon\rho} \to\ H_{\mu\upsilon\rho}/2, \eta_{\mu\upsilon} \to -\eta_{\mu\upsilon}, \eta_{IJ} \to -\eta_{IJ}, \gamma^{\mu} \to i\gamma^{\mu}$ and $\lambda \to \lambda/\sqrt{2}$. In the expression for the potential given in eq. (2.19) of [138], $\mathcal{A}^{A}_{\alpha\,B}\mathcal{A}^{B}_{\beta\,A}\xi^{\alpha i}\xi^{\alpha j}$ should be replaced by $\left(\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i}-\ \delta^{A}_{B}\right)\left(\mathcal{A}^{B}_{\beta\,A}\xi^{i} - \delta^{B}_{A}\right)$, and in eq. (2.20) for $\delta\lambda^{iA}$, the expression $\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i}$ should be replaced by $\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i} - \delta^{A}_{B}$

[18] 我们通过令 $c^{I}_{z} \to -c^{I}_{z}/2, H_{\mu\upsilon\rho} \to\ H_{\mu\upsilon\rho}/2, \eta_{\mu\upsilon} \to -\eta_{\mu\upsilon}, \eta_{IJ} \to -\eta_{IJ}, \gamma^{\mu} \to i\gamma^{\mu}$ 和 $\lambda \to \lambda/\sqrt{2}$, 将文献 [137] 的约定转换为文献 [136] 的约定。在文献 [138] 式 (2.19) 给出的势的表达式中，$\mathcal{A}^{A}_{\alpha\,B}\mathcal{A}^{B}_{\beta\,A}\xi^{\alpha i}\xi^{\alpha j}$ 应替换为 $\left(\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i}-\ \delta^{A}_{B}\right)\left(\mathcal{A}^{B}_{\beta\,A}\xi^{i} - \delta^{B}_{A}\right)$; 在 $\delta\lambda^{iA}$ 的式 (2.20) 中, 表达式 $\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i}$ 应替换为 $\mathcal{A}^{A}_{\alpha\,B}\xi^{\alpha i} - \delta^{A}_{B}$

The positivity of the Yang-Mills kinetic term requires that $c^{I}_{z}L_{I} > 0$, which must hold at least in a region of the moduli space, if the theory is to be physically acceptable. As to the significance of the point at which it vanishes as a phase transition point, see [139].

杨-米尔斯动能项为正要求 $c^{I}_{z}L_{I} > 0$ ，若该理论物理自洽，该条件至少在模空间的某个区域内必须成立。关于该量 vanishing 处作为相变点的意义，参见文献 [139]。

The Yang-Mills field equation resulting from the action above, upon the use of the duality equation (115), is now given by

利用对偶方程 (115)，由上述作用量得到的杨-米尔斯场方程如下

$$D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=\frac{1}{12}\varepsilon^{\upsilon\rho\sigma\lambda\tau\kappa}\eta_{IJ}c_z^I\left(H_{\rho\sigma\lambda}F^z_{\tau\kappa}-\frac{1}{2}c^J_{z'}F^{z'}_{\rho\sigma}\omega^{z'}_{\lambda\tau\kappa}-\frac{3}{4}c^{Jz'}A_\rho\mathrm{tr}_{z'}\left(F_{\sigma\lambda}F_{\tau\kappa}\right)\right).$$

(121)

Writing this equation as $D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=\hat{J}_z^\upsilon$, and taking the covariant divergence of both sides, one now finds the (consistent) anomaly [135]

将该方程写为 $D_\mu\left(c^{Iz}L_IF_z^{\mu\upsilon}\right)=\hat{J}_z^\upsilon$ ，并对两边取协变散度，即可得到 (自洽的) 反常 [135]

$$D_\mu\hat{J}_z^\mu=\frac{1}{16}\varepsilon^{\mu\upsilon\rho\sigma\lambda\tau}\eta_{IJ}c_z^Ic^I_{z'}\left(\partial_\mu A_\upsilon\right)\mathrm{tr}_{z'}\left(F_{\rho\sigma}F_{\lambda\tau}\right). \tag{122}$$

For a description of the resulting supersymmetry anomaly, and how it satisfies the Wess-Zumino consistency conditions, see [135]. At the one-loop level, generically there will be gauge, gravitational, and mixed anomalies. An analysis of the effective action including one loop effects, and potential counterterms, is needed to determine the fate of these anomalies. Global anomalies can also arise. Much work has been done on anomalies in $(1,0)$, $6D$ supergravities, and the review of this vast subject goes beyond the scope of this survey. See for example, [132, 140, 141] and the references therein.

关于由此得到的超对称反常，以及它如何满足维斯-朱米诺自洽条件，参见文献 [135]。在单圈层面，一般会存在规范反常、引力反常和混合反常。要确定这些反常的性质，需要分析包含单圈效应和可能抵消项的有效作用量。也可能出现整体反常。目前学界对 $(1,0)$, $6D$ 超引力中的反常已有大量研究，对这一广阔主题的综述超出了本文的讨论范围，相关例子参见文献 [132, 140, 141] 及其中的参考文献。

Turning to the supertransformations of the fermionic fields in leading order in fermion terms, which are given by

接下来讨论费米子场领头阶的超变换，其形式如下

$$\delta\psi_\mu=D_\mu\varepsilon+\frac{1}{48}L_IH^I_{\rho\sigma\tau}\gamma^{\rho\sigma\tau}\gamma_\mu\varepsilon$$

$$\delta\chi^r=\frac{1}{2}\left(L_I{}^r\partial_\mu L^I\right)\gamma^\mu\varepsilon+\frac{1}{24}L_I{}^rH^I_{\mu\upsilon\rho}\gamma^{\mu\upsilon\rho}\varepsilon,$$

$$\delta\lambda_z=-\frac{1}{4}F_{z\mu\nu}\gamma^{\mu\nu}\varepsilon \tag{123}$$

We shall next describe special cases of the matter couplings of $(1,0)$ supergravity known as the magical supergravities.

我们接下来介绍 $(1,0)$ 超引力物质耦合的特殊情形，即所谓的神奇超引力。

(1, 0) Magical Supergravities in 6D

六维 (1, 0) 魔法超引力

There exists a special class of supergravity theories in $D = 3, 4, 5, 6$, known as magical supergravities [142, 143] whose remarkable geometries and symmetries correspond to those of the Magic Square of Freudenthal, Rozenfeld and Tits. The scalar manifolds arising in all magical supergravities are collected are tabulated in Appendix C. The magical theories in $6D$ are parent theories from which all magical supergravities in $D = 3, 4, 5$ can be obtained by dimensional reduction, see Table 2 below. The geometries arising in $D = 3, 4, 5$ [142] were later referred to as very special quaternionic Kähler, very special Kähler, and very special real, respectively. See [144] for a review of these geometries, their relation to $6D$ theories, and a more complete list of references. Stringy origins and constructions of some of the magical supergravity theories in various dimensions, with or without additional hypermultiplet couplings, are known [145-147]. Gaugings of magical supergravities have been investigated in $D = 5$ (see [148] and the references therein) as well as in 4 and 3 dimensions [149-152], and finally in $6D$ [153], where coupling to hypermultiplets was constructed as well. Here we shall briefly review the key results of [153].

$D = 3, 4, 5, 6$ 中存在一类特殊的超引力理论，称为魔法超引力 [142, 143]，其非凡的几何与对称性对应弗罗因德塔尔、罗森菲尔德和蒂茨的魔方。所有魔法超引力的标量流形已整理汇总于附录 C。$6D$ 的魔法理论是母理论，$D = 3, 4, 5$ 的所有魔法超引力都可通过维数约化从中得到，见下文表 2。$D = 3, 4, 5$ 中得到的几何 [142] 后来分别被称为非常特殊四元 Kähler 几何、非常特殊 Kähler 几何和非常特殊实几何。关于这些几何、它们与 $6D$ 理论的关系以及更完整的参考文献列表，可参见综述 [144]。人们已经知道不同维度下部分魔法超引力理论的弦论起源与构造，无论是否存在额外超多重态耦合 [145-147]。魔法超引力的定规已在 $D = 5$ (见 [148] 及其中的参考文献)、4 维和 3 维 [149-152] 得到研究，最终在 $6D$ [153] 完成，其中也构造了与超多重态的耦合。本文我们将简要综述 [153] 的核心结果。

In $(1, 0)$, $6D$ magical supergravities, the global symmetry groups, representation content of the vectors and two-forms are displayed in Table 3. Note that the global symmetry for $n_T = 5$ and $n_T = 3$ has additional factor $USp(2)$ and $U(1)$, respectively, exclusively acting on the hypermultiplets. The structure of magical supergravity Lagrangian is in many ways similar to the one described in the previous section but it also differs in interesting ways, in particular relying on the existence of the gamma-matrix identity,

在 $(1, 0)$, $6D$ 魔法超引力中，整体对称群、向量和二形式的表示内容列于表 3。注意 $n_T = 5$ 和 $n_T = 3$ 的整体对称分别额外包含因子 $USp(2)$ 和 $U(1)$ ，它们仅作用于超多重态。魔法超引力拉氏量的结构在很多方面与上一节描述的类似，但也存在值得关注的差异，尤其依赖 gamma 矩阵恒等式的存在，

$$\Gamma_{I(AB}\Gamma^{I}_{C)D} = 0. \tag{124}$$

Given that there is a fixed number of vector fields in magical supergravities, the embedding tensor formalism is the most convenient method for determining their gaugings. In this framework, the general covariant derivative is

由于魔法超引力中的向量场数目固定，嵌入张量形式是确定其定规最简便的方法。在此框架下，一般协变导数为

$$D_\mu = \partial_\mu - A_\mu^A X_A \tag{125}$$

where the representation carried by the vector field is as given in Table 3, and

其中向量场携带的表示如表 3 所示，且

$$X_A = \theta_A{}^{IJ} t_{IJ} + \theta_A{}^{\chi} t_{\chi} + \Theta_A{}^{\mathcal{A}} t_{\mathcal{A}}, \tag{126}$$

Table 2 Scalar target spaces of magical supergravities in 6, 5, 4 and 3 dimensions

表 2 6、5、4 和 3 维魔法超引力的标量靶空间

$D=6$	$D=5$ $D=4$	$D=3$
$\frac{SO(9,1)}{SO(9)} \rightarrow$	$\frac{E_{6(-26)}}{F_4} \rightarrow \frac{E_{7(-25)}}{E_6 \times SO(2)}$	$\rightarrow \frac{E_{8(-24)}}{E_7 \times SU(2)}$
$\frac{SO(5,1)}{SO(5)} \rightarrow$	$\frac{SU^*(6)}{USp(6)} \rightarrow \frac{SO^*(12)}{U(6)}$	$\rightarrow \frac{E_{7(-5)}}{SO(12)\times SU(2)}$
$\frac{SO(3,1)}{SO(3)} \rightarrow$	$\frac{SL(3,\mathbb{C})}{SO(3)} \rightarrow \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\rightarrow \frac{E_{6(+2)}}{SU(6)\times SU(2)}$
$\frac{SO(2,1)}{SO(2)} \rightarrow$	$\frac{SL(3,\mathbb{R})}{SO(3)} \rightarrow \frac{Sp(6,\mathbb{R})}{U(3)}$	$\rightarrow \frac{F_{4(+4)}}{USp(6)\times USp(2)}$

Table 3 The first column shows the full global symmetry groups of magical supergravities, the second column gives the representation content of the vector fields under these groups, whose reality properties are listed in the third column: Majorana (M), Weyl (W), Majorana-Weyl (MW), symplectic Majorana-Weyl (SMW). The last column gives the two-form representation content

表 3 第一列给出魔法超引力的完整整体对称群，第二列给出向量场在这些群下的表示内容，第三列列出表示的实性质: 马约拉纳 (M)、外尔 (W)、马约拉纳-外尔 (MW)、辛马约拉纳-外尔 (SMW)。最后一列给出二形式的表示内容

G_T	Vectors	Vectors reality	Γ^I_{AB}	Two-forms
$SO(9,1)$	16_c	MW	Γ^I_{AB}	10
$SO(5,1)\times USp(2)$	$(4_c,2)$	SMW, $A=(\alpha r)$	$\Gamma^I_{\alpha r,\beta s} = \Gamma^I_{\alpha\beta}\varepsilon_{rs}$	$(6,1)$
$SO(3,1)\times U(1)$	$(2,1)_+ + (1,2)_-$	W, $A=\{\alpha,\dot\beta\}$	$\begin{pmatrix} 0 & \Gamma^I_{\alpha\dot\beta} \\ \bar\Gamma^I_{\dot\alpha\beta} & 0 \end{pmatrix}$	$(2,2)_0$
$SO(2,1)$	2	M	Γ^I_{AB}	3

where t_{IJ} are the generators of $SO(n_T,1)$, the generators t_χ span the additional symmetries $USp(2)$ and $U(1)$ for $n_T=5$ and $n_T=3$, respectively, and $t_{\mathcal{A}}$ are generators of the isometries of the quaternionic Kähler manifold. [19] The generators X_A are required to obey the closed algebra

其中 t_{IJ} 是 $SO(n_T,1)$ 的生成元，生成元 t_χ 张成 $n_T=5$ 和 $n_T=3$ 分别对应的额外对称性 $USp(2)$ 和 $U(1)$ ，$t_{\mathcal{A}}$ 是四元 Kähler 流形等距的生成元。[19] 生成元 X_A 需要满足封闭代数

$$[X_A, X_B] = -X_{AB}{}^C X_C,\ X_{(AB)}{}^C X_C = 0. \tag{127}$$

As shown in [153], the consistency of gauge algebra on vector fields imposes the constraint

如 [153] 所示，规范代数在向量场上的自洽性施加了约束

$$X_{(BC)}{}^{A} = \Gamma^{I}_{BC}\theta^{A}_{I}, \tag{128}$$

where θ^{A}_{I} is a constant tensor. Consequently, the hierarchy of the covariant field strength takes the form

其中 θ^{A}_{I} 是常数张量。因此，协变场强的分层形式为

$$\begin{aligned}
\mathcal{F}^{A}_{\mu\upsilon} &= 2\partial_{[\mu}A^{A}_{\upsilon]} + X_{[BC]}{}^{A}A^{B}_{\mu}A^{C}_{\upsilon} + B^{I}_{\mu\upsilon}\theta^{A}_{I}, \\
\mathcal{H}^{I}_{\mu\upsilon\rho} &= 3D_{[\mu}B^{I}_{\upsilon\rho]} + 6\Gamma^{I}_{AB}A_{[\mu}{}^{A}\left(\partial_{\upsilon}A_{\rho]}{}^{B} + \frac{1}{3}X_{[CD]}{}^{B}A_{\upsilon}{}^{C}A_{\rho]}{}^{D}\right) + \theta^{IA}C_{\mu\upsilon\rho A}, \\
\mathcal{G}_{\mu\upsilon\rho\sigma A} &= 4D_{[\mu}C_{\upsilon\rho\sigma]A} - \Gamma_{IAB}\left(6B^{I}_{[\mu\upsilon}\mathcal{F}^{B}_{\rho\sigma]} + 6\theta^{BJ}B^{I}_{[\mu\upsilon}B_{\rho\sigma]J}\right. \\
&\left. +8\Gamma^{I}_{CD}A^{B}_{[\mu}A^{C}_{\upsilon}\partial_{\rho}A^{D}_{\sigma]} + 2\Gamma^{I}_{CD}X_{EF}{}^{D}A_{[\mu}{}^{B}A_{\upsilon}{}^{C}A_{\rho}{}^{E}A_{\sigma]}{}^{F}\right).
\end{aligned} \tag{129}$$

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[19] Note that the R -symmetry group $Sp(1)_R$ is contained in $t_{\mathcal{A}}$.

[19] 注意到 R 对称群 $Sp(1)_R$ 包含在 $t_{\mathcal{A}}$ 中。

The gauge transformations and the Bianchi identities can be found in [153]. The construction so far is entirely off-shell, but the (anti-) self-duality condition (115) will be imposed by hand, and the field equation of $B^{I}_{\mu\nu}$, up to fermion terms, obtained from the pseudo-action will give rise to the first order equation

规范变换和比安基恒等式可在 [153] 中找到。迄今为止的构造完全是 off-shell 的，但需要手动加上 (反) 自对偶条件 (115)，从准作用量得到的 $B^{I}_{\mu\nu}$ 场方程 (忽略费米子项) 会给出一阶方程

$$\theta^{A}_{I}\left(\mathcal{G}^{\mu\nu\rho\sigma}_{A} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}m_{AB}\mathcal{F}^{B}_{\lambda\tau}\right) = 0, \tag{130}$$

where

其中

$$m_{AB} \equiv L_{I}\Gamma^{I}_{AB}. \tag{131}$$

The constraints on the embedding tensor, linear and quadratic, turn out to be highly constraining, and it was shown that they are solved such that

嵌入张量的线性和二次约束约束性极强，可以证明约束的解满足

$$\theta^{IJ}_{A} = -\Gamma^{[I}_{AB}\theta^{J]B}, \; \theta^{IA} = \Gamma^{I}_{BC}\zeta^{A}\zeta^{B}\zeta^{C}, \; \theta^{IA}\Theta_{A}{}^{\mathcal{A}} = 0, \tag{132}$$

where ζ^A is an unconstrained constant (commuting) spinor of $SO(n_T, 1)$. All different choices of the spinor ζ^A lead to equivalent gaugings [153].

其中 ζ^A 是 $SO(n_T, 1)$ 的一个无约束常数 (对易) 旋量。所有对旋量 ζ^A 的不同选择都会得到等价的规范定域 [153]。

In the hypermultiplet sector, the generators $t_{\mathcal{A}}$ act on the hyperscalar manifold by a Killing vector field $K_{\mathcal{A}}{}^X$ as

在超多重态部分，生成元 $t_{\mathcal{A}}$ 通过 Killing 向量场 $K_{\mathcal{A}}{}^X$ 作用在超标量流形上，形式如下

$$t_{\mathcal{A}} \cdot \phi^X = K_{\mathcal{A}}^X(\phi) \tag{133}$$

and the gauge covariant derivative of the hyperscalars is given by

超标量的规范协变导数由下式给出

$$D_\mu \phi^X = \partial_\mu \phi^X - g A_\mu^A \mathcal{K}_A^X,\ \mathcal{K}_A^X \equiv \theta_A{}^{\mathcal{A}} K_{\mathcal{A}}^X. \tag{134}$$

The covariant derivative of the fermionic fields requires the following connections

费米子场的协变导数需要以下联络

$$Q_\mu^{ab} = L^{I[a} D_\mu L_I^{b]},\ Q_\mu^{ij} = \partial_\mu \phi^X + A_\mu^A C_A{}^{ij},\ Q_\mu^{rs} = \partial_\mu \phi^X + A_\mu^A C_A{}^{rs},$$

(135)

where

其中

$$C_{Ai}{}^j = -\frac{1}{n_H} V_{ri}^X V_Y^{rj} \nabla_X \mathcal{K}_A^Y,\ C_{Ar}{}^s = -\frac{1}{2} V_{ri}^X V_Y^{si} \nabla_X \mathcal{K}_A^Y. \tag{136}$$

Putting together the above ingredients, and the following the standard Noether procedure, we find that the pseudo-action for the gauged magical supergravity is given by

整合上述要素后，遵循标准的诺特 procedure，我们得到定域规范神奇超引力的赝作用量为

$$e^{-1}\mathcal{L} = R - \frac{1}{12} g_{IJ} \mathcal{H}_{\mu\nu\rho}^I \mathcal{H}^{\mu\nu\rho J} - \frac{1}{4} m_{AB} \mathcal{F}_{\mu\nu}^A \mathcal{F}^{\mu\nu B} - \frac{1}{4} P_\mu^a P^{\mu a} - \frac{1}{2} P_\mu^{ri} P_{\mu ri}$$

$$-\frac{1}{4}\left(\theta^{IA}\theta^{JB} m_{AB} g_{IJ} + m^{AB} C_{Aij} C_B^{ij}\right) + \mathcal{L}_{\text{top}}, \tag{137}$$

where $\mathcal{L}_{\text{top}}$ is the gauge invariant completion of the $B \wedge \mathcal{F} \wedge \mathcal{F}$ term and its variation is given in [153, eq. (4.42)]. Using that variation, it is easy to see the equation of motion for $B_{\mu\nu}^I$, up to fermion terms, gives the (projected) duality equation (130). The supertransformations of the fermions terms are

其中 $\mathcal{L}_{\text{top}}$ 是 $B\wedge\mathcal{F}\wedge\mathcal{F}$ 项的规范不变完备化，其变分可见文献 [153, 式 (4.42)]。利用该变分不难看出，忽略费米子项后，$B^I_{\mu\nu}$ 的运动方程给出 (投影) 对偶方程 (130)。费米子项的超变换为

$$\delta\psi^i_\mu = \mathcal{D}_\mu\varepsilon^i + \frac{1}{48}\gamma^{\rho\sigma\tau}\gamma_\mu\varepsilon^i\mathcal{H}_{\rho\sigma\tau}$$

$$\delta\chi^a_i = \frac{1}{2}\gamma^\mu\varepsilon_i P^a_\mu - \frac{1}{24}\gamma^{\mu\upsilon\rho}\varepsilon_i\mathcal{H}^a_{\mu\upsilon\rho},$$

$$\delta\lambda^A_i = -\frac{1}{4}\gamma^{\mu\upsilon}\varepsilon_i\mathcal{F}^A_{\mu\upsilon} - \frac{1}{2}\theta^{IA}L_I\varepsilon_i - \frac{1}{2}m^{AB}C_{Bij}\varepsilon^j,$$

$$\delta\psi^r = P^{ri}_\mu\gamma^\mu\varepsilon_i$$

where $\mathcal{H}_{\mu\upsilon\rho} := \mathcal{H}^I_{\mu\upsilon\rho}L_I$ and $\mathcal{H}^a_{\mu\upsilon\rho} := \mathcal{H}^I_{\mu\upsilon\rho}L_I{}^a$. In establishing the supersymme-try of the action, it is important to recall that the duality equations $\mathcal{H}^+_{\mu\upsilon\rho}=0$ and $\mathcal{H}^{a-}_{\mu\upsilon\rho}=0$ are to be used after varying action.

其中 $\mathcal{H}_{\mu\upsilon\rho} := \mathcal{H}^I_{\mu\upsilon\rho}L_I$ 和 $\mathcal{H}^a_{\mu\upsilon\rho} := \mathcal{H}^I_{\mu\upsilon\rho}L_I{}^a$ 。在证明作用量的超对称性时，需要注意对偶方程 $\mathcal{H}^+_{\mu\upsilon\rho}=0$ 和 $\mathcal{H}^{a-}_{\mu\upsilon\rho}=0$ 需要在对作用量变分后使用。

To describe the gauge group, consider the case of $n_T=9$, from which all the lower magical theories can be obtained by truncation. In this case, there exists the 3-graded decomposition $so(9,1)\rightarrow N^-_{(8)}\oplus so(8)\oplus so(1,1)\oplus N^+_{(8)}$, where the subscripts refer to the $so(1,1)$ charges, and $N^\pm_{(8)}$ are nilpotent generators. Accordingly, splitting the spinor index of $so(9,1)$ as $A=(\alpha,t,0)$, where $\alpha = 1,\dots,8, t=1,\dots,7$, in this basis the gauge algebra is found to be non-semisimple nilpotent with commutation rules taking the form

为描述规范群，考虑 $n_T=9$ 的情形，所有更低阶的神奇理论都可以通过该情形截断得到。该情形下存在 3-分次分解 $so(9,1)\rightarrow N^-_{(8)}\oplus so(8)\oplus so(1,1)\oplus N^+_{(8)}$ ，其中下标对应 $so(1,1)$ 电荷，$N^\pm_{(8)}$ 是幂零生成元。相应地，将 $so(9,1)$ 的旋量指标拆分为 $A=(\alpha,t,0)$ ，其中 $\alpha = 1,\dots,8, t=1,\dots,7$ ，在此基底下规范代数被证明为非半单幂零代数，对易关系形式为

$$\left[X_\alpha,X_\beta\right] = -g\gamma^t_{\alpha\beta}X_t,\ [X_\alpha,X_t]=0=[X_t,X_u]\,,\ X_0=0. \tag{138}$$

The generators X^t act as central extensions of the algebra which vanish when evaluated on vector and tensor fields but can act nontrivially on the hypermultiplet scalars. More specifically, $X_\alpha\phi^X = \theta^{IJ}_\alpha t_{IJ}\phi^X + \Theta_\alpha{}^{\mathcal{A}}K^X_{\mathcal{A}}$ and $X_t\phi^X = \Theta_t{}^{\mathcal{A}}K^X_{\mathcal{A}}$, where ϕ^X denotes the hypermultiplet scalars, and $K^X_{\mathcal{A}}$ are the Killing vectors supported by the quaternionic Kähler manifold generating an algebra isomorphic to the one in (138). For various embeddings of this group into the isometries of the hyperscalar manifolds, and further details, see [153].

生成元 X^t 对应代数的中心扩张，它们在矢量场和张量场上作用为零，但可以在超多重态标量上有非平凡作用。更具体地，$X_\alpha\phi^X = \theta^{IJ}_\alpha t_{IJ}\phi^X + \Theta_\alpha{}^{\mathcal{A}}K^X_{\mathcal{A}}$ 和 $X_t\phi^X = \Theta_t{}^{\mathcal{A}}K^X_{\mathcal{A}}$ ，其中 ϕ^X 表示超多重态标量，$K^X_{\mathcal{A}}$ 是四元数凯勒流形上的 Killing 向量，它们生成的代数同构于 (138) 中的代数。关于该群嵌入超标量流形等距群的各类情形以及更多细节，可见文献 [153]。

In the case of generic couplings of Yang-Mills and multi-tensor multiplets, the Yang-Mills field equations $D_\mu F^{\mu\upsilon} = J^\mu$ displays a gauge anomaly in the form of $D_\mu J^\mu = \mathcal{A}$. In the case of magic supergravities whose tensor-Yang-Mills content is fixed as given in Table 3, one finds that $D_\mu J^\mu = 0$ by using the gamma-matrix identity (124). This holds also in presence of the hypermultiplet couplings. Nonetheless, only in the ungauged theory, and by introducing $n_H = 273 + n_V - 29 n_T$ hypermultiplets, and in the case of $n_T = 2, 3, 5$ by employing Green-Schwarz-Sagnotti mechanism [66, 154] as well, that the gravitational anomaly can be removed. In presence of gaugings, gravitational, gauge, and mixed anomalies are generically expected to arise at the quantum level. Their analysis and possible removal remains to be investigated.

对于杨-米尔斯多重态与多张量多重态的一般耦合情形，杨-米尔斯场方程 $D_\mu F^{\mu\nu} = J^\mu$ 会呈现出形式为 $D_\mu J^\mu = \mathcal{A}$ 的规范反常。对于表 3 中给出的张量-杨-米尔斯内容已固定的魔法超引力，利用伽马矩阵恒等式 (124) 可得到 $D_\mu J^\mu = 0$ ，该结论在包含超多重态耦合时仍然成立。尽管如此，只有在未规范理论中，且引入 $n_H = 273 + n_V - 29 n_T$ 超多重态，并在 $n_T = 2, 3, 5$ 的情况下额外采用格林-施瓦茨-萨尼奥蒂机制 [66, 154]，才能消除引力反常。存在规范作用时，一般认为量子层面会产生引力反常、规范反常和混合反常，对它们的分析与反常消除仍有待研究。

Comments on Other Supergravities in 6D

六维其他超引力评论

(3, 1) and (4, 0) Exotic Supergravities and Anomalous (2, 1) Supergravity

(3, 1) 与 (4, 0) 奇异超引力及反常 (2, 1) 超引力

In $6D$, in addition to the $(2,2)$ Poincaré superalgebra, the $(3,1)$, $(4,0)$ and $(2,1)$ versions also exist. In the first two cases, the lowest dimensional representations do not contain the graviton, and hence the terminology of "exotic supergravities." Such supergravities have been conjectured to exist and to describe the strong coupling limits of maximal supergravity in $5D$ [155, 156] . Evidence has been presented for an exceptional field theory which can possibly accommodate these theories, or the one with $(2, 2)$ supersymmetry, depending on solutions of the embedding constraints. However, nonlinear theories for exotic supergravities have not been constructed as yet.

在 $6D$ 中，除了 $(2,2)$ 庞加莱超代数外，还存在 $(3,1)$, $(4,0)$ 和 $(2,1)$ 版本。在前两种情况中，最低维表示不包含引力子，因此得名“奇异超引力”。这类超引力被猜想存在，并用来描述 $5D$ [155, 156] 中极大超引力的强耦合极限。已有研究证明存在例外场论，根据嵌入约束解的不同，它可以容纳这些理论，也可以容纳具有 $(2, 2)$ 超对称的理论。但目前奇异超引力的非线性理论仍未构建完成。

In the case of $(3,1)$, the R -symmetry group is $USp(6) \times USp(2)$ and the lowest dimensional representation has the field content

在 $(3,1)$ 的情形中，R 对称群是 $USp(6) \times USp(2)$ ，最低维表示的场内容为

$$(3,1): \ \{C_{\mu\nu,\rho}, B^{ai}_{\mu\nu+}, A^{ab}_\mu, \phi^{abi}; \psi^a_{\mu+}, \psi^i_{\mu\nu-}, \chi^{ab}_+, \chi^{abi}_-\}, \tag{139}$$

where $a = 1, \ldots, 6$ and $i = 1, 2$ label the fundamentals of $USp(6)$ and $USp(2)$), the pairs of $USp(6)$ indices are symplectic traceless, and the exotic field has the hook symmetry, $C_{\mu\nu,\rho} = -C_{\upsilon\mu,\rho}, C_{[\mu\upsilon,\rho]} = 0$. The two-form bosonic field, and the exotic two-form fermion, have self-dual field strengths, and $\pm$ indicate chiralities of the fermions. The field equations for this multiplet exhibit curious features [157, 158].

其中 $a = 1, \ldots, 6$ 和 $i = 1, 2$ 分别标记 $USp(6)$ 和 $USp(2)$ 的基础表示，成对的 $USp(6)$ 指标是辛无迹的，奇异场具有钩对称性，即 $C_{\mu\nu,\rho} = -C_{\upsilon\mu,\rho}, C_{[\mu\upsilon,\rho]} = 0$ 。二形式玻色场和奇异二形式费米子具有自对偶场强，$\pm$ 标记费米子的手征性。该多重态的场方程展现出奇特的性质 [157, 158]。

In the case of $(4, 0)$, the R -symmetry group is $USp(8)$ and the lowest dimensional representation has the field content

在 $(4, 0)$ 的情形中，R 对称群是 $USp(8)$ ，最低维表示的场内容为

$$(4,0): \ \{C_{\mu\upsilon,\rho\sigma}, B^{ab}_{\mu\upsilon+}, \phi^{abcd}; \psi^{a}_{\mu+}, \chi^{abc}_{-}\}, \tag{140}$$

where $a = 1, \ldots, 8$ labels the $USp(8)$ fundamental, the exotic bosonic field has the window symmetry, $C_{\mu\upsilon,\rho\sigma} = C_{\rho\sigma,\mu\upsilon} = -C_{\upsilon\mu,\rho\sigma}, C_{[\mu\upsilon,\rho\sigma]} = 0$. The two-form boson and spin 1/2 fermions are in the 27 and 42 dimensional representations of $Sp(4)$, respectively. As usual, $\pm$ indicate chiralities of the fermions. The yet to be constructed interacting $(4, 0)$ theory has been conjectured to have a global $E_{6(6)}$ symmetry with scalars parametrizing the $E_{6(6)}/USp(8)$ coset [155]. Based on its field content, the (4, 0) theory has gravitational anomalies, which, however, can be cancelled by adding 21 tensor multiplets [159, 160]. For further interesting aspects of $(4, 0)$, $6D$ supergravity, see [161,162].

其中 $a = 1, \ldots, 8$ 标记 $USp(8)$ 基础表示，奇异玻色场具有窗口对称性，即 $C_{\mu\upsilon,\rho\sigma} = C_{\rho\sigma,\mu\upsilon} = -C_{\upsilon\mu,\rho\sigma}, C_{[\mu\upsilon,\rho\sigma]} = 0$ 。二形式玻色子和自旋 1/2 费米子分别属于 $Sp(4)$ 的 27 维和 42 维表示。和惯例一样，$\pm$ 标记费米子的手征性。尚未构建完成的相互作用 $(4, 0)$ 理论被猜想具有整体 $E_{6(6)}$ 对称性，其中标量参数化 $E_{6(6)}/USp(8)$ 陪集 [155]。基于其场内容，$(4, 0)$ 理论存在引力反常，但该反常可以通过添加 21 个张量多重态抵消 [159, 160]。关于 $(4, 0)$, $6D$ 超引力的更多有趣性质，参见 [161,162]。

Finally, let us recall that there also exists a $(2, 1)$ multiplet with the R symmetry group $Sp(2) \times Sp(1)$ that has the field content

最后，我们回顾一下，还存在一个 $(2, 1)$ 多重态，其具有 R 对称群 $Sp(2) \times Sp(1)$ ，场内容为

$$(2,1): \{e^{a}_{\mu},\ B_{\mu\nu+},\ B_{\mu\nu-},\ A_{\mu},\ \phi,\ \psi_{\mu+},\ \psi_{\mu-},\ \chi_{+},\ \chi_{-}\},$$

$$(1,1)\ \ (1,1)\ \ (4,1)\ \ (4,2)\ \ (5,1)\ \ (1,2)\ \ (4,2)\ \ (5,2)\ \ (4,1)$$

(141)

where (p, q) in the second row denote the $USp(6) \times USp(2)$ representation content. Supergravity with this field content has been studied in [160, 163]. The five scalars parametrize the coset $SO(5, 1)/SO(5)$ [164], and the theory has a bosonic sector which is identical to that of $(1, 0)$ supergravity coupled to eight vector multiplets and five tensor multiplets. These are referred to as twin supergravities [165], as their bosonic sectors coincide, while their fermionic sectors differ. Both have gravitational anomalies [163] but while $(1, 0)$

supergravity can be coupled to vector, tensor, and hypermultiplets to cure its anomalies (see next section), (2, 1) supergravity does not have such matter multiplets available to remove its anomalies. [20]

> 第二行中的 (p,q) 表示 $USp(6)\times USp(2)$ 表示内容。具有该场内容的超引力已在文献 [160, 163] 中研究。五个标量参数化了陪集 $SO(5,1)/SO(5)$ [164]，该理论的玻色子部分与耦合了八个矢量多重态和五个张量多重态的 $(1,0)$ 超引力的玻色子部分完全相同。它们被称为孪生超引力 [165]，因为二者玻色子部分一致，但费米子部分不同。二者都存在引力反常 [163]，但 $(1,0)$ 超引力可以通过耦合矢量、张量和超多重态消除反常 (见下一节)，而 $(2,1)$ 超引力没有可用的这类物质多重态来消除自身的反常。[20]

D = 5

We shall summarize the results for maximal $N = 8$ supergravity, half-maximal $N = 4$ supergravity coupled to vector multiplets, and $N = 2$ supergravity coupled to vector, tensor and hypermultiplets. In the last case, some of the vectors, the number of which depends on the gauged symmetries, are dualized to two-form potentials. The gaugings in the embedding tensor formalism will be summarized for $N = 8,4$. The scalar manifold geometries involved are tabulated in Appendix C.

> 我们将总结极大 $N=8$ 超引力、耦合矢量多重态的半极大 $N=4$ 超引力，以及耦合矢量、张量和超多重态的 $N=2$ 超引力的研究结果。在最后一种情况中，矢量的数量取决于定域规范对称性，其中部分矢量被对偶化为二形式势。我们将针对 $N=8,4$ 总结嵌入张量形式体系中的定域规范。涉及的标量流形几何已整理列入附录 C 的表格中。

$N = 8$ Gauged Supergravity in 5D

> ## $N=8$ 五维定域规范超引力

Maximal supergravity 5*D* was constructed long ago in [166], and certain gaugings of it were obtained in [167, 168]. Using the embedding tensor formalism, the most general gaugings were constructed in [169], and it is this work which we summarize briefly below. In the embedding formalism framework, which brings in a tensor hierarchy, the maximal gauged supergravity is built out of the following multiplet of fields

> 极大超引力 5*D* 早在文献 [166] 中就已构造完成，部分定域规范形式也在文献 [167, 168] 中得到。利用嵌入张量形式，最一般的定域规范构造由文献 [169] 完成，下文我们将对该工作做简要总结。在引入张量层次结构的嵌入形式框架下，极大定域规范超引力由以下场多重态构造而成

$$\left(e_\mu{}^m, \mathcal{V}_M{}^{ij}, A_\mu^M, B_{\mu\nu M}; \psi_\mu^i, \chi^{ijk}\right), \tag{142}$$

where $i = 1,\ldots,8$ labels the fundamental representation of the R -symmetry group $USp(8)$, $\mathcal{V}_M{}^{ij}$ is the $E_{6(6)}/USp(8)$ coset representative, and $M = 1,\ldots 27$ labels the fundamental representation of $E_{6(6)}$. The spinors are symplectic Majorana, $\chi^{ijk} = \chi^{[ijk]}$ and $\Omega_{ij}\chi^{ijk} = 0$, thus in the 48-plet of $USp(8)$. The two-form potential $B_{\mu\nu M}$, in 27 of $E_{6(6)}$, is introduced as on-shell dual of the vector fields A_μ^M . Their covariantized field strengths will be related to each or via a duality equation, which arises, under a projection, as an equation of motion. All of the 27 gauge fields, or a subset of them, may be used to gauge a suitable subgroup of $E_{6(6)}$. This gauging is encoded in a real embedding tensor ${\theta_M}^{\alpha}$ which determines the gauge group G_0 among the $E_{6(6)}$ generators t_α with $\alpha = 1, \dots, 78$, as

其中 $i = 1, \dots, 8$ 标记 R 对称群的基础表示, $USp(8)$, ${\mathcal{V}_M}^{ij}$ 是 $E_{6(6)}/USp(8)$ 陪集代表元, $M = 1, \dots 27$ 标记 $E_{6(6)}$ 的基础表示。旋量为辛马约拉纳旋量 $\chi^{ijk} = \chi^{[ijk]}$ 和 $\Omega_{ij}\chi^{ijk} = 0$ ，因此属于 $USp(8)$ 的 48 维表示。二形式势 $B_{\mu\upsilon M}$ 属于 $E_{6(6)}$ 的 27 维表示，它是作为矢量场 A_μ^M 的在壳对偶引入的。它们的协变场强通过对偶方程相互关联，该方程是投影后的运动方程。全部 27 个规范场或其子集可用于给 $E_{6(6)}$ 的合适子群定规范，这种定规范由实嵌入张量 ${\theta_M}^{\alpha}$ 编码，它在满足 $\alpha = 1, \dots, 78$ 的 $E_{6(6)}$ 生成元 t_α 中确定规范群 G_0 ，即

$$X_M = {\theta_M}^{\alpha} t_\alpha \tag{143}$$

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[20] Adding one gravitino multiplet of suitable chirality removes this anomaly, but result is just the (2, 2) theory.

[20] 添加一个手性合适的引力微子多重态即可消除该反常，最终得到的就是 (2, 2) 理论。

Thus, the covariant derivatives are given by $D_\mu = \nabla_\mu - gA_\mu^M X_M$. Supersymmetry requirement imposes a linear constraint on the embedding tensor such that in the product **27 ⊗ 78** , only the representations351survives. This implies the conditions

因此，协变导数由 $D_\mu = \nabla_\mu - gA_\mu^M X_M$ 给出。超对称要求对嵌入张量施加线性约束: 在乘积 **27 ⊗ 78** 中仅保留 351 表示，由此得到条件

$$t_{\alpha M}{}^{N} {\theta_N}^{\alpha} = 0, \ \left(t_\beta t_\alpha\right)_M{}^{N} {\theta_N}^{\beta} = 0. \tag{144}$$

The closure of gauge algebra, on the other hand, imposes the quadratic constraints

另一方面，规范代数的封闭性要求满足二次约束

$$f_{\beta\gamma}{}^{\alpha} {\theta_M}^{\beta} {\theta_N}^{\gamma} + t_{\beta N}{}^{P} {\theta_M}^{\beta} {\theta_P}^{\alpha} = 0. \tag{145}$$

As a consequence, it is shown in [169] that

因此，文献 [169] 中证明了

$$X_{(MN)}{}^{P} = d_{MNQ} Z^{PQ}, \ X_{[MN]}{}^{Q} = 10 d_{MQS} d_{NRT} d^{PQR} Z^{ST}, \tag{146}$$

where d_{MNP} is the totally symmetric invariant tensor of E_6 and $Z_{MN} = Z_{[MN]}$ is defined by these relations. In the scalar field sector, the $E_{6(6)}$ valued Maurer-Cartan decomposes as

其中 d_{MNP} 是 E_6 的全对称不变张量，$Z_{MN} = Z_{[MN]}$ 由上述关系定义。标量场部分，取值于 $E_{6(6)}$ 的莫雷-嘉当形式可分解为

$$\mathcal{V}_{ij}{}^{M}\left(\partial_\mu \mathcal{V}_M{}^{k\ell} - gA_\mu{}^P X_{PM}{}^N \mathcal{V}_N{}^{k\ell}\right) = P_{\mu ij}{}^{k\ell} + 2Q_{\mu[i}{}^{[k}\delta_{j]}^{\ell]}. \tag{147}$$

In the conventions of [169], $[X_M, X_N] = X_{MN}{}^P X_P$. The bosonic Lagrangian is given by [169]

在文献 [169] 的约定下，$[X_M, X_N] = X_{MN}{}^P X_P$ 。玻色子拉格朗日量由文献 [169] 给出为

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{16}\mathcal{F}^{ij}_{\mu\nu}\mathcal{F}^{\mu\nu}_{ij} - \frac{1}{12}\left|P^{ijk\ell}_\mu\right|^2 + 3g^2\left|A^{ij}_1\right|^2 - \frac{1}{3}g^2\left|A^{i,jk\ell}_2\right|^2$$

$$+e^{-1}\mathcal{L}_{\text{top}} \tag{148}$$

where

其中

$$\mathcal{F}^{ij}_{\mu\upsilon} = \mathcal{V}^{ij}_M\left(2\partial_{[\mu}A^M_{\upsilon]} + gX_{[NP]}{}^M A_\mu{}^N A_\upsilon{}^P + gZ^{MN}B_{\mu\upsilon N}\right), \tag{149}$$

and $\left(A^{ij}_1, A^{ijk\ell}_2\right)$ are defined by

且 $\left(A^{ij}_1, A^{ijk\ell}_2\right)$ 由下式定义

$$X_{MN}{}^P \mathcal{V}_P{}^{[k\ell]}\mathcal{V}^{mn]N}\mathcal{V}_{ij}{}^M = 12A_2^{q,[k\ell m]}\delta^{n]}{}_{[i}\Omega_{j]q} + 9A_2^{p,q[k\ell}\Omega^{mn]}\Omega_{p[i}\Omega_{j]q},$$

$$X_{MN}{}^P \mathcal{V}_{Pim}\mathcal{V}^{jmN}\mathcal{V}^{k\ell M} = 3\Omega_{im}A_2^{(m,j)k\ell} + 3\Omega_{im}\left(\Omega^{m[k}A_1^{\ell]j} + \Omega^{j[k}A_1^{\ell]m}\right.$$

$$\left.+\frac{1}{4}\Omega^{k\ell}A_1^{mj}\right) \tag{150}$$

and $\mathcal{L}_{\text{top}}$, which has a complicated form but simple general variation, can be found explicitly in [169]. Using that variation, it is straightforward to find that the field equation of $B_{\mu\upsilon M}$ is, modulo the fermion terms, the (projected) duality equation [169]

而 $\mathcal{L}_{\text{top}}$ 形式复杂但广义变分形式简单，其显式表达式可在文献 [169] 中找到。利用该变分可以很容易得到，扣除费米子项后 $B_{\mu\upsilon M}$ 的场方程就是 (投影后的) 对偶方程 [169]

$$Z^{MN}\left(\mathcal{H}^{\mu\upsilon\rho}{}_N - \frac{i}{\sqrt{5}}\varepsilon^{\mu\upsilon\rho\sigma\tau}\mathcal{M}_{NP}\mathcal{F}^P_{\sigma\tau}\right) = 0, \tag{151}$$

where the "internal" metric $\mathcal{M}_{MN} = \mathcal{V}_M{}^{ij}\mathcal{V}_{Nij}$, and the field strength $\mathcal{H}_{\mu\upsilon\rho M}$ is defined by $3D_{[\mu}\mathcal{F}^M_{\upsilon\rho]} = gZ^{MN}\mathcal{H}_{\mu\upsilon\rho N}$.

其中“内禀”度规为 $\mathcal{M}_{MN} = \mathcal{V}_M{}^{ij}\mathcal{V}_{Nij}$ ，场强 $\mathcal{H}_{\mu\upsilon\rho M}$ 由 $3D_{[\mu}\mathcal{F}^M_{\upsilon\rho]} = gZ^{MN}\mathcal{H}_{\mu\upsilon\rho N}$ 定义。

Finally, the supertransformations are given by [169]

最后，超变换由文献 [169] 给出

$$\delta\psi^i_\mu = D_\mu\varepsilon^i - \frac{1}{12}i\left(\gamma_\mu{}^{\upsilon\rho} - 4\delta^\upsilon_\mu\gamma^\rho\right)\mathcal{F}_{\upsilon\rho}{}^{ij}\varepsilon_j + igA_1{}^{ij}\varepsilon_j,$$

$$\delta\chi^{ijk} = -\frac{1}{2}i\gamma^\mu P^{ijk\ell}_\mu\varepsilon_\ell - \frac{3}{16}\gamma^{\mu\upsilon}\left(\mathcal{F}_{\mu\upsilon}{}^{[ij}\varepsilon^{k]} + \frac{1}{3}\Omega^{[ij}\mathcal{F}_{\mu\upsilon}{}^{k]\ell}\varepsilon_\ell\right) - gA_2^{\ell,ijk}\varepsilon_\ell,$$

(152)

where $D_\mu\varepsilon^i = \left(\partial_\mu\delta^i_j - \frac{1}{4}\omega_{\mu ab}\gamma_{ab}\delta^i_j - Q_{\mu j}{}^i\right)\varepsilon^j$.

其中 $D_\mu\varepsilon^i = \left(\partial_\mu\delta^i_j - \frac{1}{4}\omega_{\mu ab}\gamma_{ab}\delta^i_j - Q_{\mu j}{}^i\right)\varepsilon^j$ 。

$N = 6$ Supergravity in 5D

$N = 6$ 五维超引力

The field content of $N = 6$ supergravity is

$N = 6$ 超引力的场内容为

$$\left\{e^r_\mu, A^{ij}_\mu, A_\mu, V^{ij}_\alpha; \psi^i_\mu, \chi^i, \chi^{ijk}\right\}, \tag{153}$$

where $i = 1, \ldots, 6$ is an $USp(6)$ index, the fermions are symplectic Majorana, A^{ij}_μ and χ^{ijk} are in14and **14**′ representations of $USp(6)$ and the scalars parametrize the coset $SU^\star(6)/USp(6)$ [170]. Ungauged or gauged, the Lagrangians and supersymmetry transformations of the $N = 6$ theory have not been spelled out entirely in the literature so far. However, it is straightforward to obtain them from a consistent reduction of the $N = 8$ theory, which requires the removal of a single gravitino multiplet.

其中 $i = 1, \ldots, 6$ 为 $USp(6)$ 指标，费米子是辛马约拉纳费米子，A^{ij}_μ 和 χ^{ijk} 属于 $USp(6)$ 的 14 维和 **14**′ 表示，标量参数化陪集 $SU^\star(6)/USp(6)$ [170]。无论是未规范还是规范情形，目前文献中尚未完整给出 $N = 6$ 理论的拉格朗日量和超对称变换。不过可以通过对 $N = 8$ 理论做一致约化直接得到这些结果，约化过程只需移除一个引力微子多重态。

The $N = 6, 5D$ supergravity has the same bosonic sector as that of $N = 2, 5D$ supergravity, where the coset space parametrized by the scalars, $SU^\star(6)/USp(6)$, arises as one of the very special real (VSR) manifolds in the magic family. As such they are referred to as ”twins.” We shall say more on this in section ” $N = 6$ Super-gravity in $4D$ ”. For various other aspects of the $N = 6, 5D$ supergravity, see [171].

$N=6,5D$ 超引力的玻色子部分与 $N=2,5D$ 超引力相同,由标量参数化的陪集空间 $SU^{\star}(6)/USp(6)$ 是魔法族中非常特殊实流形 (VSR) 的一员，因此二者被称为“孪生”超引力。我们会在“$4D$ 维中的 $N=6$ 超引力”一节详细讨论。关于 $N=6,5D$ 超引力的其他更多方面，参见文献 [171]。

$N=4$ Supergravity Coupled to Vector Multiplets in 5D

$N=4$ 超引力与矢量多重态耦合 (5 维)

We combine the $N=4$ supergravity multiplet with n copies of the vector multiplet $\left(A_\mu, 5\phi; \lambda\right)$ and introduce the notation

我们将 $N=4$ 超引力多重态与 n 份矢量多重态 $\left(A_\mu, 5\phi; \lambda\right)$ 结合，并引入如下记号

$$\left\{e_\mu^r, A_\mu^M, \phi, \mathcal{V}_\alpha{}^A; \psi_\mu^i, \chi^i, \lambda^{ia}\right\},\ A_\mu^M=\left\{A_\mu^0, A_\mu^\alpha\right\}, \tag{154}$$

where $\alpha=1,\dots,n+5, a=1,\dots,n$ and $\mathcal{V}_\alpha{}^A=(\mathcal{V}_\alpha{}^m, \mathcal{V}_\alpha{}^a)$ with $m=\ 1,\dots,5$, is a representative of the coset $SO(n,5)/SO(n)\times SO(5)$ parametrized by the scalar fields, and ϕ is the dilaton. The vectors $\left(A_\mu^0, A_\mu^m\right)$ belong to the supergravity multiplet. Note that the abelian vector gauge fields form one vector A_μ^α and one singlet A_μ^0 under $SO(n,5)$. The index $i=1,\dots,4$ labels the fundamental representation of $USp(4)_R\approx SO(5)$, and the fermions are symplectic Majorana. The undeformed theory has the global symmetry $SO(1,1)\times SO(n,5)$. The gauging of (on-shell) trombone symmetry will not be considered here. For gauging of such symmetries, see [8].

其中,由标量场参数化陪集 $SO(n,5)/SO(n)\times SO(5)$ 的代表元是带 $m=\ 1,\dots,5$ 的 $\alpha=1,\dots,n+5, a=1,\dots,n$ 和 $\mathcal{V}_\alpha{}^A=(\mathcal{V}_\alpha{}^m, \mathcal{V}_\alpha{}^a)$, ϕ 是 dilaton(伸缩子)。矢量 $\left(A_\mu^0, A_\mu^m\right)$ 属于超引力多重态。注意，阿贝尔矢量规范场在 $SO(n,5)$ 下形成一个矢量 A_μ^α 和一个单态 A_μ^0 。指标 $i=1,\dots,4$ 标记 $USp(4)_R\approx SO(5)$ 的基础表示，费米子是辛马约拉纳费米子。未形变理论具有整体对称性 $SO(1,1)\times SO(n,5)$ 。本文不讨论 (壳) 长号对称性的规范作用，这类对称性的规范作用可参见文献 [8]。

Pure $N=4$ supergravity was constructed by Noether procedure in [107] where a subset of the vector fields were used to gauge $SU(2)\times U(1)$. Coupling n vector multiplets to supergravity, in which a particular $SU(2)$ is gauged, was obtained in [172]. A more general gauging with the gauge group taking the form $K_A\times K_S$ where K_A is an abelian and K_S is a semisimple subgroup of $SO(n,5)$ was achieved in [173], where it was shown that K_A has to be $SO(2)$ or $SO(1,1)$ gauged by a_μ . These results were generalized further in [82] to account for most general gaugings, using the embedding tensor formalism, which we summarize next. The $SO(5,n)$ invariant tensor $\eta=\mathrm{diag}(-1,-1,-1,-1,-1,+1,\dots,+1)$ and the positive definite scalar matrix $\mathcal{M}$ are given by

纯 $N=4$ 超引力由诺特定程序在文献 [107] 中构造完成，其中部分矢量场被用来对 $SU(2)\times U(1)$ 做规范作用。将 n 个矢量多重态耦合到超引力 (其中对特定 $SU(2)$ 做了规范作用) 的结果在文献 [172] 中得到。规范群形式为 $K_A\times K_S$ (其中 K_A 是阿贝尔子群, K_S 是 $SO(n,5)$ 的半单李子群) 的更一般规范作用在文献 [173] 中实现，该文证明 K_A 必须是由 a_μ 做规范作用的 $SO(2)$ 或 $SO(1,1)$ 。这些结果在文献 [82] 中利用嵌入张量形式进一步推广到最一般的规范作用，我们接下来对此做总结。$SO(5,n)$ 不变张量 $\eta=\mathrm{diag}(-1,-1,-1,-1,-1,+1,\dots,+1)$ 和正定标量矩阵 $\mathcal{M}$ 由下式给出:

$$\eta_{\alpha\beta}=-\mathcal{V}_\alpha{}^m\mathcal{V}_\beta{}^m+\mathcal{V}_\alpha{}^a\mathcal{V}_\beta{}^a,\ \mathcal{M}_{\alpha\beta}=\mathcal{V}_\alpha{}^m\mathcal{V}_\beta{}^m+\mathcal{V}_\alpha{}^a\mathcal{V}_\beta{}^a. \tag{155}$$

We also define $\mathcal{V}_\alpha^{ij}\equiv\mathcal{V}_\alpha{}^m(\Gamma_m)^{ij}$ where $\Gamma_m^{ij}=\Gamma_m^{[ij]}$ are the $SO(5)$ gamma-matrices. [21] Taking into account the fact that A_μ^α and A_μ^0 have $SO(1,1)$ charges 1/2 and -1, respectively, the gauge group generators $(X_M)_N{}^P$ in the covariant derivative $D_\mu=\nabla_\mu+gA_\mu^MX_M$, are parametrized as [82]

我们还定义了 $\mathcal{V}_\alpha^{ij}\equiv\mathcal{V}_\alpha{}^m(\Gamma_m)^{ij}$ ，其中 $\Gamma_m^{ij}=\Gamma_m^{[ij]}$ 是 $SO(5)$ 伽马矩阵。[21] 考虑到 A_μ^α 和 A_μ^0 分别带 $SO(1,1)$ 电荷 1/2 和-1，协变导数 $D_\mu=\nabla_\mu+gA_\mu^MX_M$ 中的规范群生成元 $(X_M)_N{}^P$ 可参数化为 [82]

$$X_{\alpha\beta}{}^\gamma=-f_{\alpha\beta}{}^\gamma-\frac{1}{2}\eta_{\alpha\beta}\xi^\gamma+\delta^\gamma_{[\alpha}\xi_{\beta]},\ X_{\alpha0}{}^0=\xi_\alpha,\ X_{0\alpha}{}^\beta=\xi_\alpha{}^\beta. \tag{156}$$

Thus the general gauging is parametrized by three real tensors, $f_{\alpha\beta\gamma},\xi_{\alpha\beta}$ and ξ_α . The closure of the gauge algebra imposes the following quadratic constraints [82]

因此，一般规范作用由三个实张量 $f_{\alpha\beta\gamma},\xi_{\alpha\beta}$ 和 ξ_α 参数化。规范代数的封闭性给出下列二次约束 [82]

$$\xi_\alpha\xi^\alpha=0,\ \xi_{\alpha\beta}\xi^\beta=0,\ f_{\alpha\beta\gamma}\xi^\gamma=0,$$

$$3f_{\kappa[\alpha\beta}f_{\gamma\delta]}{}^\kappa=2f_{[\alpha\beta\gamma}\xi_{\delta]},\ \xi_\alpha{}^\kappa f_{\kappa\beta\gamma}=\xi_\alpha\xi_{\beta\gamma}-\xi_{[\beta}\xi_{\gamma]\alpha}. \tag{157}$$

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[21] The conventions are: $\Omega^{ij}\Omega_{jk}=-\delta^i_k$, and $X^i=\Omega^{ik}X_k$ and $X_i=X^k\Omega_{ki}$.

[21] 约定如下: $\Omega^{ij}\Omega_{jk}=-\delta^i_k$ 、$X^i=\Omega^{ik}X_k$ 和 $X_i=X^k\Omega_{ki}$ 。

The covariant field strengths are given by

协变场强由下式给出

$$\mathcal{F}^M_{\mu\nu}=2\partial_{[\mu}A^M_{\nu]}+gX_{NP}{}^MA^N_\mu A_\nu{}^P+gZ^{MN}B_{\mu\nu N}, \tag{158}$$

$$Z^{MN}\mathcal{H}_{\mu\nu\rho N}=Z^{MN}\left[3D_{[\mu}B_{\nu\rho]N}+6gd_{NPQ}A^P_{[\mu}\left(\partial_\nu A^Q_\rho\right)+\frac{1}{3}gX_{RS}{}^QA^R_\nu A^S_{\rho]})\right],$$

where $d_{MNP}=d_{(MNP)}$ and $Z_{MN}=Z_{[AB]}$ are defined by

其中 $d_{MNP} = d_{(MNP)}$ 和 $Z_{MN} = Z_{[AB]}$ 定义为

$$d_{0\alpha\beta} = d_{\alpha 0\beta} = d_{\alpha\beta 0} = \eta_{\alpha\beta}, \text{ all other components zero,}$$

$$Z^{\alpha\beta} = \frac{1}{2}\xi^{\alpha\beta},\ Z^{0\alpha} = -Z^{\alpha 0} = \frac{1}{2}\zeta^{\alpha}. \tag{159}$$

We will also need the coset current and composite $SO(5) \approx USp(4)$ connection

我们还需要陪集流和复合 $SO(5) \approx USp(4)$ 联络

$$\mathcal{P}_{\mu}^{aij} = \mathcal{V}^{\alpha a} D_{\mu} \mathcal{V}_{\alpha}{}^{ij},\ \mathcal{Q}_{\mu i}{}^{j} = 2\mathcal{V}_{ik}^{\alpha} D_{\mu} \mathcal{V}_{\alpha}{}^{jk}. \tag{160}$$

We are now ready to give the Lagrangian [82]

现在我们可以给出拉格朗日量 [82]

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{4}e^{-\phi}\mathcal{M}_{\alpha\beta}\mathcal{F}_{\mu\nu}^{\alpha}\mathcal{F}^{\mu\nu\beta} - \frac{1}{4}e^{2\phi}\mathcal{F}_{\mu\nu}^{0}\mathcal{F}^{0\mu\nu} - \frac{3}{8}\partial_{\mu}\phi\partial^{\mu}\phi + \mathcal{P}_{\mu}^{aij}\mathcal{P}_{aij}^{\mu}$$

$$-4\left(A_1^{ij}A_{1ij} - A_2^{ij}A_{2ij} - A_2^{aij}A_{2ij}^{a}\right) + e^{-1}\mathcal{L}_{\text{top}}, \tag{161}$$

where the shift functions are given by

其中平移函数由下式给出

$$A_1^{ij} = \frac{1}{\sqrt{6}}\left(-\zeta^{ij} + 2\rho^{ij}\right),\ A_2^{ij} = \frac{1}{\sqrt{6}}\left(\zeta^{ij} + \rho^{ij} + \frac{3}{2}\tau^{ij}\right),$$

$$A_2^{aij} = \frac{1}{2}\left(-\zeta^{aij} + \rho^{aij} - \frac{\sqrt{2}}{4}\tau^{a}\Omega^{ij}\right), \tag{162}$$

with the scalar field dependent tensors defined as

依赖标量场的张量定义为

$$\tau^{ij} = e^{\phi/2}\mathcal{V}_{\alpha}{}^{ij}\zeta^{\alpha},\ \tau^{a} = e^{\phi/2}\mathcal{V}_{\alpha}{}^{a}\zeta^{\alpha},$$

$$\zeta^{ij} = \sqrt{2}e^{-\phi}\mathcal{V}_{\alpha}{}^{i}{}_{k}\mathcal{V}_{\beta}{}^{jk}\zeta^{\alpha\beta},\ \zeta^{aij} = e^{\phi}\mathcal{V}_{\alpha}{}^{a}\mathcal{V}_{\beta}{}^{ij}\zeta^{\alpha\beta}, \tag{163}$$

$$\rho^{ij} = -\frac{2}{3}e^{\phi/2}\mathcal{V}_{\alpha}^{ik}\mathcal{V}_{\beta}^{j\ell}\mathcal{V}^{\gamma}{}_{k\ell}f^{\alpha\beta}{}_{\gamma},\ \rho^{aij} = \sqrt{2}e^{\phi/2}\mathcal{V}_{\alpha}{}^{a}\mathcal{V}_{\beta}{}^{i}{}_{k}\mathcal{V}_{\gamma}{}^{jk}f^{\alpha\beta\gamma}.$$

The tensors $\left(\zeta^{ij}, \rho^{ij}, \rho^{aij}\right)$ are symmetric, and $\left(\lambda^{ij}, \zeta^{aij}\right)$ are anti-symmetric, and therefore $A_1^{ij} = A_1^{(ij)}$. The topological Lagrangian $\mathcal{L}_{\text{top}}$ is given in [82] where its general variation is also provided,

张量 $(\zeta^{ij}, \rho^{ij}, \rho^{aij})$ 是对称的，$(\lambda^{ij}, \zeta^{aij})$ 是反对称的，因此 $A_1^{ij} = A_1^{(ij)}$ 。拓扑拉格朗日量 $\mathcal{L}_{\text{top}}$ 见文献 [82]，其中也给出了它的一般变分，

$$\delta\mathcal{L}_{\text{top}} = \frac{1}{4\sqrt{2}} e^{\mu\upsilon\rho\sigma\lambda}\left(\frac{1}{3}Z^{MN}\mathcal{H}_{\mu\upsilon\rho M}\Delta B_{\sigma\lambda N} + d_{MNP}\mathcal{F}^M_{\mu\upsilon}\mathcal{F}^N_{\rho\sigma}\delta A^P_\lambda\right), \tag{164}$$

up to total derivatives, and with $\Delta B_{\mu\upsilon M} \equiv Z^{MN}\left(\delta B_{\mu\upsilon M} - 2d_{NPQ}A^P_{[\mu}\delta A^Q_{\upsilon]}\right)$. It follows that the equation of motion for the two-form field is the (projected) duality equation

差一个全导数项，且满足 $\Delta B_{\mu\upsilon M} \equiv Z^{MN}\left(\delta B_{\mu\upsilon M} - 2d_{NPQ}A^P_{[\mu}\delta A^Q_{\upsilon]}\right)$。由此可得二形式场的运动方程就是 (投影) 对偶方程

$$Z^{MN}\left(\frac{1}{6\sqrt{2}}\varepsilon_{\mu\nu\rho\sigma\lambda}\mathcal{H}^{\rho\sigma\lambda}{}_N - \mathcal{M}_{NP}\mathcal{F}^P_{\mu\nu}\right) = 0, \tag{165}$$

where $\mathcal{M}_{MN} \equiv \operatorname{diag}\left(e^{-4\sigma}, e^{2\sigma}M_{\alpha\beta}\right)$. The Yukawa couplings and the following supertransformations are given by [82]:

其中 $\mathcal{M}_{MN} \equiv \operatorname{diag}\left(e^{-4\sigma}, e^{2\sigma}M_{\alpha\beta}\right)$。汤川耦合和下列超变换由文献 [82] 给出:

$$\begin{aligned}
\delta\psi^i_\mu &= D_\mu\varepsilon^i + \frac{i}{6}\left(e^{-\phi/2}\mathcal{V}^{ij}_\alpha\mathcal{F}^\alpha_{\upsilon\rho} + \frac{\sqrt{2}}{4}\Omega^{ij}\mathcal{F}^0_{\upsilon\rho}\right)\left(\gamma_\mu{}^{\upsilon\rho} - 4\delta^\upsilon_\mu\gamma^\rho\right)\varepsilon_j - \frac{1}{\sqrt{6}}A_1^{ij}\gamma_\mu\varepsilon_j, \\
\delta\chi^i &= -\frac{\sqrt{3}i}{4}\partial_\mu\phi\gamma^\mu\varepsilon^i + \frac{\sqrt{3}}{6}\left(e^{-\phi/2}\mathcal{V}_M{}^{ij}\mathcal{F}^M_{\mu\nu} - e^\phi\Omega^{ij}\mathcal{F}^0_{\mu\nu}\right)\gamma^{\mu\upsilon}\varepsilon_j - \sqrt{2}A_2^{ij}\varepsilon_j, \\
\delta\lambda^{ia} &= \mathcal{P}^{aij}_\mu\gamma^\mu\varepsilon_j - \frac{1}{4}e^{-\phi/2}\mathcal{V}_M{}^a\mathcal{F}^M_{\mu\upsilon}\gamma^{\mu\upsilon}\varepsilon^i + A_2^{aij}\varepsilon_j,
\end{aligned} \tag{166}$$

where $D_\mu\varepsilon^i = D_\mu(\omega, \mathcal{Q})\varepsilon^i$.

其中 $D_\mu\varepsilon^i = D_\mu(\omega, \mathcal{Q})\varepsilon^i$ 。

$N = 2$ Supergravity Coupled to Vector, Tensor and Hypermultiplets in 5D and Very Special Real Manifolds

$N = 2$ 五维下耦合矢量、张量与超多重子的超引力及非常特殊实流形

The $N = 2, 5D$ Poincaré superalgebra in $5D$ admits the following multiplets

$N = 2, 5D$ 庞加莱超代数在 $5D$ 中允许存在以下多重态

$$\underbrace{\left(e^A_\mu, \psi^i_\mu, A_\mu\right)}_{\text{graviton}}, \ \underbrace{\left(A_\mu, \sigma, \lambda^i\right)}_{\text{vector}} \ \underbrace{\left(B_{\mu\nu}, \varphi, \chi^i\right)}_{\text{tensor}}, \ \underbrace{(4\phi, \psi)}_{\text{hyper}}. \tag{167}$$

The spinors are symplectic Majorana, $i = 1,2$ labels the doublet of the R symmetry group $USp(2)$. General couplings of two-derivative $N = 2$ supergravity in $5D$ coupled to n_V vector multiplets, n_T tensor multiplets and n_H hypermultiplets, including gaugings of possible isometries of the scalar manifolds, based largely on previous work, e.g., [143, 174, 175], was given completely in [176], where a more complete list of references can be found. Here we shall follow the book [118] to give a brief account of these couplings. The conventions are largely those of [176].

旋量是辛马约拉纳旋量，$i = 1,2$ 标记 R 对称群 $USp(2)$ 的二重态。基于前人工作 [例如 143, 174, 175]，包含标量流形可能等距的规范作用，两导数项 $N = 2$ 超引力在 $5D$ 下耦合 n_V 个矢量多重态、n_T 个张量多重态和 n_H 个超多重态的一般耦合已在文献 [176] 中被完整给出，该文献也列出了更完整的参考文献。本文我们将参考著作 [118] 对这些耦合做简要介绍，约定基本沿用文献 [176] 的设定。

To begin with, we group together field of the supergravity multiplet with those of n_V vector multiplets and n_T tensor multiplets and use the notation

首先，我们将超引力多重态的场与 n_V 个矢量多重态、n_T 个张量多重态的场归为一组，并使用记号

$$\{e_\mu^a, \psi_\mu^i, A_\mu^I, B_{\mu\nu}^r, \varphi^x; \psi_\mu^i, \lambda^{ix}\},\ \{\phi^X, \psi^A\}, \tag{168}$$

where

其中

$$I = 0, 1, \dots, n_V,\ r = 1, \dots, n_T,\ x = 1, \dots, n_V + n_T,$$

$$X = 1, \dots, 4n_H,\ A = 1, \dots, 2n_H,\ i = 1, 2. \tag{169}$$

The hyperscalars parametrize a quaternionic Kähler manifold $\mathcal{M}_{QK}$, while $n_V + n_T$ real scalars φ^x parametrize a very special real manifold $\mathcal{M}_{VSR}$. To define $\mathcal{M}_{VSR}$, introduce $n_V + n_T + 1$ scalars $h^{\tilde{I}}$ and impose the constraint

超标量参数化四元数凯勒流形 $\mathcal{M}_{QK}$ ，而 $n_V + n_T$ 个实标量 φ^x 参数化非常特殊实流形 $\mathcal{M}_{VSR}$ 。为定义 $\mathcal{M}_{VSR}$ ，引入 $n_V + n_T + 1$ 个标量 $h^{\tilde{I}}$ 并施加约束

$$C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1,\ \tilde{I} = 0, 1, \dots, n_V + n_T, \tag{170}$$

where $C_{\tilde{I}\tilde{J}\tilde{K}}$ is a totally symmetric constant real tensor. The index $\tilde{I}$ is split as $\tilde{I} = (I, r)$ with $I = 0, 1, \dots, n_V$ and $r = n_{V+1}, \dots, n_V + n_T$. The following relations fields can then be defined

其中 $C_{\tilde{I}\tilde{J}\tilde{K}}$ 是全对称常数实张量。指标 $\tilde{I}$ 拆分为 $\tilde{I} = (I, r)$ ，满足 $I = 0, 1, \dots, n_V$ 和 $r = n_{V+1}, \dots, n_V + n_T$ 。由此可以定义以下关系场

$$a_{\tilde{I}\tilde{J}} = -2C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{K}} + 3h_{\tilde{I}} h_{\tilde{J}},\ g_{xy} = a_{\tilde{I}\tilde{J}} h_x^{\tilde{I}} h_y^{\tilde{J}},$$

$$h_{\tilde{I}}h^{\tilde{I}} = 1,\ h_{\tilde{I}} = a_{\tilde{I}\tilde{J}}h^{J},\ h_x^{\tilde{I}} = -\sqrt{\frac{3}{2}}h^{\tilde{I}}_{,x}, \tag{171}$$

where $a_{\tilde{I}\tilde{J}}$ is the metric in the ambient space, and g_{xy} is the metric on $\mathcal{M}_{VSR}$. For several useful identities, see for example [143, 176]. In particular, all cubic polynomials whose invariance group acts transitively on the corresponding real spaces have been classified in [120]. The symmetric VSR manifolds are listed in Table 5, under $D = 5, N = 2$.

其中 $a_{\tilde{I}\tilde{J}}$ 是外围空间的度量，g_{xy} 是 $\mathcal{M}_{VSR}$ 上的度量。若干有用恒等式可参见例如文献 [143, 176]。特别地，不变群可传递作用在对应实空间上的所有三次多项式已在文献 [120] 中完成分类。对称非常特殊实流形 (VSR) 已列于表 5 的 $D = 5, N = 2$ 项下。

Next, consider the gauge transformations [118]

接下来，考虑规范变换 [118]

$$\delta A^{I}_{\mu} = \partial_{\mu}\Lambda^{I} - \Lambda^{J} f_{JK}{}^{I} A^{K}_{\mu},\ \delta B^{r}_{\mu\nu} = -\Lambda^{J} t_{J\tilde{I}}{}^{r} H^{\tilde{I}}_{\mu\nu},$$

$$\delta\varphi^{X} = \Lambda^{I} k_{I}{}^{X},\ \delta\phi^{x} = \Lambda^{I} k_{I}{}^{x},\ k_{I}{}^{x} := \sqrt{\frac{3}{2}} t_{I\tilde{J}}{}^{\tilde{K}} h^{\tilde{J}} h_{\tilde{K}}{}^{x}, \tag{172}$$

where $\Lambda^{I}(x)$ is the gauge parameter, $k_{I}{}^{X}(\phi)$ and $k_{I}{}^{x}(\varphi)$ are the Killing vectors on the manifolds parametrized by the scalars of the vector and hypermultiplets, respectively, and they separately generate the gauge algebra with structure constants $f_{JK}{}^{I}$. Furthermore, the generators t_{I} in the $(n_{V} + n_{T} + 1)$ dimensional representation are given by

其中 $\Lambda^{I}(x)$ 是规范参数，$k_{I}{}^{X}(\phi)$ 和 $k_{I}{}^{x}(\varphi)$ 分别是矢量多重态标量和超多重态标量参数化流形上的基灵矢量，它们分别生成带有结构常数 $f_{JK}{}^{I}$ 的规范代数。此外，$(n_{V} + n_{T} + 1)$ 维表示中的生成元 t_{I} 由下式给出

$$(t_{r})_{\tilde{J}}{}^{\tilde{K}} = 0,\ (t_{I})_{\tilde{J}}{}^{\tilde{K}} = \begin{pmatrix} f_{IJ}{}^{K} & t_{IJ}{}^{r} \\ 0 & t_{Ir}{}^{s} \end{pmatrix}, \tag{173}$$

obeying gauge algebra $[t_{I}, t_{J}] = f_{IJ}{}^{K} t_{K}$. The following relations must hold [118]

满足规范代数 $[t_{I}, t_{J}] = f_{IJ}{}^{K} t_{K}$ 。必须满足以下关系 [118]

$$C_{r\tilde{J}\tilde{K}} = \sqrt{\frac{3}{8}} t_{\tilde{J}\tilde{K}}{}^{s}\Omega_{sr},\ t_{I[r}{}^{t}\Omega_{s]t} = 0,\ t_{I(\tilde{J}}{}^{\tilde{M}} C_{\tilde{K}\tilde{L}]\tilde{M}} = 0. \tag{174}$$

With these basic ingredients at hand, the bosonic part is given by [176][22]

有了这些基本要素，玻色子部分由 [176][22] 给出

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{4}a_{\tilde{I}\tilde{J}} H^{\tilde{I}}_{\mu\nu} H^{\tilde{J}\mu\nu} - \frac{1}{2}g_{xy} D_{\mu}\varphi^{x} D^{\mu}\varphi^{y} - \frac{1}{2}g_{XY} D_{\mu}\phi^{X} D^{\mu}\phi^{Y} - V$$

$$+\frac{1}{16g}\varepsilon^{\mu\upsilon\rho\sigma\tau}B_{\mu\upsilon r}\left(\partial_\rho B^r_{\sigma\tau}+t_{Is}{}^r A^I_\rho B^s_{\sigma\tau}+2t_{IJ}{}^r A^I_\mu F^J_{\rho\sigma\tau}\right)$$

$$+\frac{1}{6\sqrt{6}}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left[F^J_{\upsilon\lambda}F^K_{\rho\sigma}+f_{FG}{}^J A^F_\upsilon A^G_\lambda\left(-\frac{1}{2}F^K_{\rho\sigma}+\frac{1}{10}f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right]$$

$$-\frac{1}{8}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{rs}t_{IK}{}^r t_{FG}{}^s A^I_\mu A^F_\upsilon A^G_\lambda\left(-\frac{1}{2}F^K_{\rho\sigma}+\frac{1}{10}f_{HL}{}^K A^H_\rho A^L_\sigma\right), \tag{175}$$

where Ω_{rs} is the constant symplectic invariant matrix, we have set the coupling constant equal to one, and

其中 Ω_{rs} 是常数辛不变矩阵，我们已将耦合常数设为 1，且

$$H^{\tilde{I}}_{\mu\upsilon}=\left(F^I_{\mu\upsilon},B^r_{\mu\upsilon}\right),\ D_\mu\varphi^x=\partial_\mu\varphi^x-A^I_\mu k_I{}^x,$$

$$D_\mu\phi^X=\partial_\mu\phi^X-A^I_\mu k_I{}^X,\ F^I_{\mu\upsilon}=2\partial_{[\mu}A^I_{\upsilon]}+f_{JK}{}^I A^J_\mu A^K_\upsilon. \tag{176}$$

The potential is given by

势由下式给出

$$V=\mathbf{P}\cdot\mathbf{P}-\frac{1}{2}\mathbf{P}^x\cdot\mathbf{P}_x-2W_xW^x-2\mathcal{N}^{iA}\mathcal{N}_{iA}, \tag{177}$$

where

其中

$$W^x=-\frac{\sqrt{6}}{4}h^Ik_I{}^x,\mathcal{N}^{iA}=-\frac{\sqrt{6}}{4}h^Ik_I{}^X\mathcal{V}^{iA}_X,$$

$$\mathbf{P}=h^I\mathbf{P}_I,\ \mathbf{P}_x=h^I{}_x\mathbf{P}_I,\ \mathbf{P}_I=\frac{1}{4n_H}\mathbf{J}_X{}^Y\nabla_Y k_I{}^X. \tag{178}$$

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[22] In the absence of hypermultiplet couplings, and for $t_{IJ}{}^r=0$, this result agrees with that of [174] where $t_{IJ}{}^r$ was set to zero.

[22] 在不存在超多重态耦合且满足 $t_{IJ}{}^r=0$ 的情况下，该结果与 $t_{IJ}{}^r$ 被设为零的文献 [174] 的结果一致。

Here $\mathbf{J}_X{}^Y$ are the three quaternionic Kähler structures, $\mathcal{V}^{iA}_X$ is the vielbein on $\mathcal{M}_{QK}$, and the moment maps $\mathbf{P}_I$ are defined by the relation (see section 3.3.2 in [177] for a detailed discussion)

在此，$\mathbf{J}_X{}^Y$ 是三个四元数凯勒结构，$\mathcal{V}^{iA}_X$ 是 $\mathcal{M}_{QK}$ 上的标架，动量映射 $\mathbf{P}_I$ 由下述关系定义 (详细讨论参见文献 [177] 的 3.3.2 节)

$$\partial_X \mathbf{P}_I = -\frac{1}{2}\mathbf{J}_{XY} k_I^Y. \tag{179}$$

The supertransformations of the fermions are given by

费米子的超变换由下式给出

$$\delta\psi_\mu^i = D_\mu \varepsilon^i + \frac{i}{4\sqrt{6}}\left[h_{\tilde{I}} H^{\tilde{I}\upsilon\rho}\left(\gamma_{\mu\upsilon\rho} - 4g_{\mu\upsilon}\gamma_\rho\right)\varepsilon^i + 2P_j{}^i\gamma_\mu\varepsilon^j\right],$$

$$\delta\lambda^{xi} = -\frac{i}{2}\gamma^\mu D_\mu \varphi^x \varepsilon^i + \frac{1}{4}\gamma\cdot H^{\tilde{I}} h_{\tilde{I}}{}^x \varepsilon^i + P^x{}_j{}^i\varepsilon^j + \frac{1}{2}W^x\varepsilon^i,$$

$$\delta\psi^A = \frac{i}{2}\gamma^\mu D_\mu \phi^X V_X^{iA}\varepsilon_i - \mathcal{N}^{iA}\varepsilon_i \tag{180}$$

where

其中

$$D_\mu \varepsilon^i = \nabla_\mu \varepsilon^i + \partial_\mu \phi^X \omega_{Xj}{}^i \varepsilon^j + \frac{1}{2}A_\mu^I P_{Ij}{}^i \varepsilon^j, \tag{181}$$

with the $USp(2)$ connection $\omega_{Xj}{}^i$ and the definition $P_{Ii}{}^j := \mathbf{P}_I \cdot (\boldsymbol{\sigma})_j{}^i$ used. The specifics of the gauge group $K \subset G$, and its representations carried by the tensor multiplets, of course, depend on the solutions of the constraints (173) and (174). In the case of last four entries for $5D, N = 2$ in the table provided in Appendix B, the following gauge groups K , and their irreps carried by the n_T tensors, are given in [174]:

这里使用了 $USp(2)$ 联络 $\omega_{Xj}{}^i$ 与定义 $P_{Ii}{}^j := \mathbf{P}_I \cdot (\boldsymbol{\sigma})_j{}^i$ 。规范群 $K \subset G$ 的具体形式，以及张量多重态承载的其表示，当然取决于约束 (173) 和 (174) 的解。对于附录 B 表格中 $5D, N = 2$ 的最后四项，文献 [174] 给出了如下规范群 K ，以及 n_T 张量承载的其不可约表示:

$$K = SO^\star(6) \subset E_{6(-26)},\ n_T = 6 + 6,$$

$$K = SU(3) \times U(1)_R \subset E_{6(-26)},\ n_T = 3 + \bar{3} + 3 + \bar{3} + 3 + \bar{3},$$

$$K = SO^\star(6) \subset SU^\star(6),\ n_T = 0,$$

$$K = SU(3) \times U(1)_R \subset SU^\star(6),\ n_T = 3 + \bar{3},$$

$$K = SU(3) \times U(1)_R \subset SL(3, \mathbb{C}),\ n_T = 0,$$

$$K = SU(2) \times U(1) \times U(1)_R \subset SL(3, \mathbb{C}),\ n_T = 2 + \bar{2},$$

$$K = SL(2, \mathbb{R}) \times U(1)_R \subset SL(3, \mathbb{C}),\ n_T = 2. \tag{182}$$

For other gaugings, including the gauging of the full R -symmetry group $SU(2)_R$, see [174, 178]. It is natural to expect that these results, as well as a systematic search for all possibilities, are best obtained by employing the embedding tensor formalism.

关于其他规范化，包括完整 R 对称群 $SU(2)_R$ 的规范化，参见文献 [174, 178]。我们自然可以认为，要得到这些结果、系统搜寻所有可能的情况，采用嵌入张量形式体系是最优方法。

Finally, it is worth noting that dimensional reduction chain $5D \to 4D \to 3D$ applied to the $N = 2, 5D$ supergravity coupled to vector multiplets, with very special real (VSR) scalar manifold, and referred to as r -maps [179] and c -maps [180], produce $4D, N = 2$ supergravity with very special Kähler (VSK), and $N = 4, 4D$ supergravity with special quaternionic Kähler (SQK) manifolds, as can be seen in the tables in Appendix C:

最后值得注意的是，对耦合了矢量多重态、拥有非常特殊实 (VSR) 标量流形的 $N = 2, 5D$ 超引力应用维数约化链 $5D \to 4D \to 3D$ ，即所谓的 r 映射 [179] 和 c 映射 [180]，会得到拥有非常特殊凯勒 (VSK) 流形的 $4D, N = 2$ 超引力，以及拥有特殊四元数凯勒 (SQK) 流形的 $N = 4, 4D$ 超引力，可见附录 C 的表格:

$$\text{VSR}\,(5D) \xrightarrow{\text{r-map}} \text{VSK}\,(4D) \xrightarrow{\text{c-map}} \text{SQK}\,(3D)\ . \tag{183}$$

See, for example, the books [118, 181] for details, and further references.

详情和更多参考文献例如可参见书籍 [118, 181]。

D = 4

Gauged supergravities in $4D$ have been reviewed extensively in [182], and a full treatment of $N = 4, 4D$ supergravity has appeared recently in [183]. Their scalar manifolds and duality symmetries are tabulated in Appendix C. For an earlier review of $N = 8$ supergravity, see [184]. The case of $N = 1, 2$ has been treated in great detail in the book [181]. For completeness, we shall give a brief survey of these supergravities.

$4D$ 中的规范超引力已在文献 [182] 中得到大量综述，$N = 4, 4D$ 超引力的完整处理最近也已发表在文献 [183] 中。它们的标量流形和对偶对称性已整理在附录 C 的表格中。关于 $N = 8$ 超引力的早期综述参见文献 [184]。$N = 1, 2$ 的情况已在书籍 [181] 中得到非常详细的讨论。为完整起见，我们将对这些超引力做一个简要概述。

$N = 8$ Gauged Supergravity in 4D

$N = 8$ 四维规范超引力

The maximal supergravity in $4D$ was constructed long ago in [185]. As is well known, this theory has on-shell global $E_{7(7)}$ symmetry. The gauging of $SU(8) \subset E_{7(7)}$ was achieved in [186]. A systematic analysis of electric gaugings was provided in [187], and a formulation which accommodates magnetic duals and the attendant more general gaugings were given in an embedding tensor formalism in [188]. We shall review the last construction here. In this framework, the $N = 8$ supergravity multiplet of fields, together with the two-form fields that are on-shell related to the scalars, are

4D 维最大超引力早在文献 [185] 中就已构造完成。众所周知，该理论存在壳上整体 $E_{7(7)}$ 对称性。$SU(8) \subset E_{7(7)}$ 的规范确定在文献 [186] 中完成。文献 [187] 给出了电规范的系统分析，而文献 [188] 通过嵌入张量形式给出了一个可容纳磁对偶及相应更一般规范的表述。本文将回顾这一最终构造。在此框架下，$N = 8$ 超引力场多重态，连同与标量场满足壳上关系的二形式场，为

$$\left(e_\mu{}^m, \mathcal{V}_M{}^A, A_\mu^M, B_{\mu\nu\alpha}; \psi_\mu^i, \chi^{ijk}\right), \tag{184}$$

where $i = 1, \dots, 8$ labels the fundamental of the R -symmetry group $SU(8)$, $M = 1, \dots, 56$ labels the fundamental representation of $E_{7(7)}$, and $\mathcal{V}_M{}^A = \left(\mathcal{V}_M{}^{ij}, \mathcal{V}_{Mij}\right)$, with antisymmetry in ij , is the vielbein on the coset $E_{7(7)}/SU(8)$. The vector fields consist of electric and magnetic potentials, $A_\mu^M = \left(A_\mu^\Lambda, A_{\mu\nu\Lambda}\right)$, with no raising and lowering of the index $\Lambda = 1, \dots, 28$. The spinors are Majorana, and $\chi^{ijk} = \chi^{[ijk]}$, thus in the 56-plet of $SU(8)$. The two-form potential $B_{\mu\nu\alpha}$ is in the adjoint representation of $E_{7(7)}$. Their properly covariantized field strengths will be related to the scalar current by a duality relation, which arises, under a projection, as an equation of motion. All of the 56 gauge fields, or a subset of them, may be used gauge a suitable subgroup of $E_{7(7)}$. This gauging is encoded in a real embedding tensor $\theta_M{}^\alpha$ which determines the gauge group G_0 among $E_{7(7)}$ generators t_α with $\alpha = 1, \dots, 133$, as

其中 $i = 1, \dots, 8$ 标记 R 对称群的基础表示，$SU(8)$, $M = 1, \dots, 56$ 标记 $E_{7(7)}$ 的基础表示，ij 取反对称的 $\mathcal{V}_M{}^A = \left(\mathcal{V}_M{}^{ij}, \mathcal{V}_{Mij}\right)$ 是陪集 $E_{7(7)}/SU(8)$ 上的标架。矢量场由电势和磁势 $A_\mu^M = \left(A_\mu^\Lambda, A_{\mu\nu\Lambda}\right)$ 组成，不对指标 $\Lambda = 1, \dots, 28$ 做升降操作。旋量是马约拉纳旋量 $\chi^{ijk} = \chi^{[ijk]}$ ，因此属于 $SU(8)$ 的 56 重态。二形式势 $B_{\mu\nu\alpha}$ 属于 $E_{7(7)}$ 的伴随表示。它们经恰当协变处理后的场强通过对偶关系与标量流相联系，该关系在投影下作为运动方程出现。全部 56 个规范场，或其中一个子集，可用于对 $E_{7(7)}$ 的合适子群做规范，该规范由实嵌入张量 $\theta_M{}^\alpha$ 编码，它在 $E_{7(7)}$ 的生成元 t_α (满足 $\alpha = 1, \dots, 133$) 中确定了规范群 G_0 ，即

$$X_M = \theta_M{}^\alpha t_\alpha \tag{185}$$

Thus, the covariant derivatives are given by $D_\mu = \nabla_\mu - g A_\mu^M X_M$. Supersymmetry requirement imposes a linear constraint on the embedding tensor such that in the product $\mathbf{56} \otimes \mathbf{133}$, only the representation912survives. This implies the conditions

因此，协变导数由 $D_\mu = \nabla_\mu - g A_\mu^M X_M$ 给出。超对称要求对嵌入张量施加一个线性约束，使得在乘积 $\mathbf{56} \otimes \mathbf{133}$ 中只有表示 912 保留下来。这给出条件

$$t_{\alpha M}{}^N \theta_N{}^\alpha = 0, \ \left(t_\beta t_\alpha\right)_M{}^N \theta_N{}^\beta = -\frac{1}{2} \theta_M{}^\alpha. \tag{186}$$

The closure of the gauge algebra, on the other hand, imposes the quadratic constraints equivalent to

另一方面，规范代数的闭合性给出二次约束，等价于

$$\Omega^{MN} \theta_M{}^\alpha \theta_N{}^\beta = 0, \tag{187}$$

provided that the constraints (186) are used. Here Ω^{IJ} is the E_7 invariant constant tensor satisfying $\Omega^{MN}\Omega_{NP} = -\delta_P^M$. There are only two $N = 8$ supersymmetric solutions of the full set of constraints on the embedding tensor. One of them is the $SU(8)$ gauged supergravity constructed long ago [186], and the other one is the so-called ω -deformed $N=8$ supergravity which was realized in [189]. We are not aware of a complete classification of all solutions to the embedding constraints. However, all electric gaugings have been found, albeit by different methods, in [187, 190].

在约束条件 (186) 成立的前提下成立。此处 Ω^{IJ} 是满足 $\Omega^{MN}\Omega_{NP}=-\delta_P^M$ 的 E_7 不变常张量。嵌入张量的全部约束仅存在两个 $N=8$ 超对称解: 一个是早已构造完成的 $SU(8)$ 规范超引力 [186]，另一个是文献 [189] 中得到的所谓 ω 形变 $N=8$ 超引力。目前我们尚未得到嵌入约束所有解的完整分类。不过文献 [187, 190] 已经通过不同方法找到了所有电规范解。

The scalar current $P_{\mu ijk\ell}$ and the composite $SU(8)$ connection are defined as

标量流 $P_{\mu ijk\ell}$ 与复合 $SU(8)$ 联络定义如下

$$P_{\mu ijk\ell}=\mathcal{V}_{ij}{}^{M}\left(\partial_\mu \mathcal{V}_M{}^{k\ell}-gA_\mu{}^{P}X_{PM}{}^{N}\mathcal{V}_{Nk\ell}\right),$$

$$Q_{\mu i}{}^{j}=\frac{2}{3}i\mathcal{V}_{ik}{}^{M}\left(\partial_\mu \mathcal{V}_M{}^{jk}-gA_\mu{}^{P}X_{PM}{}^{N}\mathcal{V}_N{}^{kj}\right), \tag{188}$$

and the bosonic part of the gauged maximal supergravity Lagrangian is given by

且带规范最大超引力拉格朗日量的玻色子部分为

$$e^{-1}\mathcal{L}=-\frac{1}{2}R-\frac{1}{4}\left(i\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\mu\nu\Sigma}+\text{ h.c. }\right)-\frac{1}{12}\left|P_\mu^{ijk\ell}\right|^2$$

$$+\frac{3}{4}g^2\left|A_1^{ij}\right|^2-\frac{1}{24}g^2\left|A_{2i}{}^{jk\ell}\right|^2+e^{-1}\mathcal{L}_{\text{ top}}, \tag{189}$$

where a 56-plet of E_7 is decomposed as $V^M=\left(V^\Lambda, V_\Lambda\right)$ with no raising and lowering of the index $\Lambda=1,\ldots,28$, and

其中 E_7 的 56 重态可分解为 $V^M=\left(V^\Lambda, V_\Lambda\right)$， 不升降指标 $\Lambda=1,\ldots,28$， 且

$$\mathcal{F}^M_{\mu\nu}=2\partial_{[\mu}A^M_{\nu]}+gX_{[NP]}{}^{M}A_\mu{}^{N}A_\nu{}^{P}+gZ^{M\alpha}B_\alpha,\ Z^{M\alpha}:=\frac{1}{2}\Omega^{MN}\theta_N{}^{\alpha}.$$

(190)

The complex (anti)self-dual projections $F^{\pm}_{\mu\nu}$ are normalized such that $F_{\mu\nu}=F^+_{\mu\nu}+F^-_{\mu\nu}$. Furthermore $A_1^{ij}, A_2^{ijk\ell}, \mathcal{N}_{\Lambda\Sigma}$ are defined by

复 (反) 自对偶投影 $F^{\pm}_{\mu\nu}$ 满足归一化条件 $F_{\mu\nu}=F^+_{\mu\nu}+F^-_{\mu\nu}$。此外 $A_1^{ij}, A_2^{ijk\ell}, \mathcal{N}_{\Lambda\Sigma}$ 定义为

$$\mathcal{V}^{N}{}_{kn}X_{MN}{}^{P}\mathcal{V}_P{}^{\ell n}\mathcal{V}^{Mij}=-3A_1^{\ell[i}\delta_k^{j]}-\frac{3}{2}A_{2k}{}^{\ell ij},$$

$$\mathcal{V}^{\Sigma\, ij}\mathcal{N}_{\Lambda\Sigma}=-\mathcal{V}_\Lambda{}^{ij} \tag{191}$$

with $A_1^{ij} = A^{[ij]}, A_{2\ell}{}^{ijk} = A_{2\ell}{}^{[ijk]}$ and $A_{2k}{}^{ijk} = 0$. As to the Lagrangian $\mathcal{L}_{\text{ top}}$, it has a complicated form but a simple general variation [188],

其中 $A_1^{ij} = A^{[ij]}, A_{2\ell}{}^{ijk} = A_{2\ell}{}^{[ijk]}$ 和 $A_{2k}{}^{ijk} = 0$ 。拉格朗日量 $\mathcal{L}_{\text{ top}}$ 形式复杂，但存在简洁的一般变分 [188],

$$e^{-1}\delta\mathcal{L}_{\text{ top}} = i\mathcal{F}^{+\mu\upsilon\Lambda} D_\mu \delta A_{\upsilon\Lambda} + \frac{1}{4} ig\mathcal{F}^{+\mu\upsilon}{}_\Lambda \theta^{\lambda\alpha} \Delta B_{\mu\upsilon\Lambda} + \text{ h.c. }, \tag{192}$$

where the covariant variation $\Delta B_{\mu\upsilon\Lambda} = \delta B_{\mu\upsilon\Lambda} - 2(t^\alpha)_M{}^P \Omega_{NP} A_\mu^M \delta A_\upsilon^N$. The two-form field is related to the scalar fields by the (projected) duality equation that follows from the Lagrangian as a field equation given by [188]

其中协变变分 $\Delta B_{\mu\upsilon\Lambda} = \delta B_{\mu\upsilon\Lambda} - 2(t^\alpha)_M{}^P \Omega_{NP} A_\mu^M \delta A_\upsilon^N$ 。二形式场与标量场满足 (投影) 对偶方程，该方程是拉格朗日量导出的场方程，形如 [188]

$$\theta^{\Lambda\alpha} \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\nu\rho\sigma\alpha} = 2iP^\Lambda{}_{ijk\ell} P^{\mu ijk\ell}, \tag{193}$$

where $P^\Lambda{}_{ijk\ell} = i\mathcal{V}^M{}_{ij} X^\Lambda{}_{MN} \mathcal{V}^N{}_{k\ell}$, and the field strength $\mathcal{H}_{\mu\upsilon\rho\alpha}$ is defined by $3D_{[\mu}\mathcal{F}_{\upsilon\rho]}^M = gZ^{M\alpha} H_{\mu\upsilon\rho\alpha}$. Finally, the supertransformations of the fermions are

其中 $P^\Lambda{}_{ijk\ell} = i\mathcal{V}^M{}_{ij} X^\Lambda{}_{MN} \mathcal{V}^N{}_{k\ell}$ ，场强 $\mathcal{H}_{\mu\upsilon\rho\alpha}$ 定义为 $3D_{[\mu}\mathcal{F}_{\upsilon\rho]}^M = gZ^{M\alpha} H_{\mu\upsilon\rho\alpha}$ 。最后，费米子的超变换为

$$\begin{aligned} \delta\psi_\mu^i &= 2D_\mu\varepsilon^i - \frac{1}{2\sqrt{2}} \mathcal{F}_{\rho\sigma}{}^{ij} \gamma^{\rho\sigma} \gamma_\mu \varepsilon_j + \sqrt{2} g A_1^{ij} \gamma_\mu \varepsilon_j, \\ \delta\chi^{ijk} &= -2\sqrt{2} P_\mu^{ijk\ell} \gamma^\mu \varepsilon_\ell + \frac{3}{2} \mathcal{F}_{\mu\upsilon}^{-}{}^{[ij} \gamma^{\mu\upsilon} \varepsilon^{k]} - 2g A_\ell{}^{ijk} \varepsilon^\ell, \end{aligned} \tag{194}$$

where $D_\mu\varepsilon^i = \left(\partial_\mu \delta_j^i - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\delta_j^i + \frac{1}{2}Q_\mu{}^i{}_j\right)\varepsilon^j$.

其中 $D_\mu\varepsilon^i = \left(\partial_\mu \delta_j^i - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\delta_j^i + \frac{1}{2}Q_\mu{}^i{}_j\right)\varepsilon^j$ 。

$N = 6$ Supergravity in 4D

$N = 6$ 四维超引力

The ungauged $N = 6$ theory can be obtained as a consistent truncation of maximal supergravity by branching $E_{7(7)} \to SO^\star(12)$ and $SU(8) \to U(6) \times SU(2)$, and retaining only the $SU(2)$ singlets [165,182,191]. The resulting field content, consisting of $64_B + 64_F$ degrees of freedom, is given by

无规范 $N = 6$ 理论可通过约化 $E_{7(7)} \to SO^\star(12)$ 和 $SU(8) \to U(6) \times SU(2)$ 、仅保留 $SU(2)$ 单态从最大超引力得到一致截断 [165,182,191]，最终得到的场内容包含 $64_B + 64_F$ 个自由度，形式为

$$\left(e_\mu{}^m, \mathcal{V}_M{}^A, A_\mu^M, B_{\mu\nu\alpha}; \psi_{\mu i}, \chi_{ijk}, \chi_i\right), \tag{195}$$

where $i = 1, \ldots, 6$ labels the fundamental representation of the R -symmetry group $SU(5) \times U(1)$, $M, A = 1, \ldots 32$ labels the $32_{\mathbf{c}}$ of $SO^\star(12)$, and $\mathcal{V}_M{}^A$ is the $G/H = SO^\star(12)/U(6)$ coset vielbein. The vectors consist of 16 electric, and 16 magnetic ones, and the duality symmetry G is realized on-shell. The spinors are Weyl, the gravitino has $U(1)$ charge $+\frac{1}{2}$, the dilatinos $\chi_{ijk} = \chi_{[ijk]}$ and χ_i are in the $20 + 6$ of H with the same chirality, and $U(1)$ charges $+\frac{3}{2}$ and -$\frac{5}{2}$, respectively. The two-form potential $B_{\mu\nu\alpha}$ is in the adjoint representation of $SO^\star(12)$. Its properly covariantized field strength will be related to the scalar current by a suitable duality relation. Various gaugings were considered in [191]. The most general gauging will be encoded as usual in an embedding tensor $\theta_M{}^\alpha$, where $\alpha = 1, \ldots, 66$. The linear constraint on the embedding tensor requires that in the product $\mathfrak{z}\mathfrak{z}_{\mathrm{c}} \times \mathbf{66}$, only the representation $\mathbf{352}_{\mathrm{s}}$ survives [165,182]. As for the quadratic constraints on the embedding tensor, it turns out that they set to zero the representations $66 + 2079' + 462_{\mathrm{s}}$ in the symmetric product $(352_{\mathrm{s}} \times 352_{\mathrm{s}})_s$ [165] .

其中 $i = 1, \ldots, 6$ 标记 R 对称群的基础表示，$SU(5) \times U(1)$, $M, A = 1, \ldots 32$ 标记 $SO^\star(12)$ 的 $32_{\mathbf{c}}$, $\mathcal{V}_M{}^A$ 是 $G/H = SO^\star(12)/U(6)$ 陪集标架。矢量包含 16 个电矢量和 16 个磁矢量，对偶对称性 G 在壳实现。旋子为外尔旋子，引力微子带 $U(1)$ 荷 $+\frac{1}{2}$ ，dilatinos $\chi_{ijk} = \chi_{[ijk]}$ 和 χ_i 属于 H 的 20+6 表示，手征性相同，$U(1)$ 荷分别为 $+\frac{3}{2}$ 和-$\frac{5}{2}$ 。二形式势 $B_{\mu\nu\alpha}$ 属于 $SO^\star(12)$ 的伴随表示。其正确协变的场强会通过合适的对偶关系与标量流关联。文献 [191] 中研究了多种不同的规范。最一般的规范和通常一样由嵌入张量 $\theta_M{}^\alpha$ 编码，其中 $\alpha = 1, \ldots, 66$ 。对嵌入张量的线性约束要求在乘积 $\mathfrak{z}\mathfrak{z}_{\mathrm{c}} \times \mathbf{66}$ 中，只有表示 $\mathbf{352}_{\mathrm{s}}$ 保留下来 [165,182]。至于嵌入张量的二次约束，其结果是将对称乘积 $(352_{\mathrm{s}} \times 352_{\mathrm{s}})_s$ [165] 中的表示 $66 + 2079' + 462_{\mathrm{s}}$ 置零。

The $N = 6$ supergravity has some exceptional properties that have been discussed in detail in [191]. One of them is the fact that the underlying superalgebra $OSp(6 \mid 4) \times SO(2)$ has zero Cartan-Killing form defined as $\mathrm{Str}(ad_X ad_Y)$. Furthermore, the zero-center module of the [23] coincides with the supergravity multiplet. Another noteworthy property is that $N = 6$ supergravity has the same bosonic sector (but different fermionic sector) and scalar coset space as one of the $N = 2$ magical supergravities in $4D$, with the special 8 vector and 5 tensor multiplet couplings. (See section " $(1,0)$ Magical Supergravities in $6D$ " for a discussion of the magical supergravities in $D = 3, 4, 5, 6$.) This is referred to as a "dual relation" in [191], and "twins" in [165]. The two theories can be obtained from different truncations of the $N = 8$ supergravity. However, this is not the case for the gauged cases [165]. In particular, the gauged versions of these twin theories can have different potentials [165]. For more details on $N = 6$ supergravity see [165,182,191].

$N = 6$ 超引力具有一些特殊性质，这些性质已在文献 [191] 中得到详细讨论。其中之一是其基础超代数 $OSp(6 \mid 4) \times SO(2)$ 的定义为 $\mathrm{Str}(ad_X ad_Y)$ 的嘉当-基灵形式为零。此外，[23] 的零中心模与超引力多重态重合。另一个值得注意的性质是: $N = 6$ 超引力与 $4D$ 中 $N = 2$ 之一的神奇超引力具有相同的玻色子 sector(但费米子 sector 不同) 和标量陪集空间，带有特殊的 8 个矢量多重态与 5 个张量多重态耦合。(关于 $D = 3, 4, 5, 6$ 中神奇超引力的讨论参见章节“$6D$ 中 $(1,0)$ 神奇超引力”。) 这一关系在文献 [191] 中被称为“对偶关系”，在文献 [165] 中被称为“孪生”。这两个理论可由 $N = 8$ 超引力的不同截断得到，但在有规范的情形下并非如此 [165]: 具体而言，这些孪生理论的规范版本可以具有不同的势 [165]。关于 $N = 6$ 超引力的更多细节参见文献 [165,182,191]。

$N = 5$ Supergravity in 4D

$N = 5$ 四维超引力

The $N = 5$ gauged supergravity does not have matter multiplets, and its field content, which has $32_B + 32_F$ degrees of freedom, is given by

$N = 5$ 规范超引力不存在物质多重态，其场内容具有 $32_B + 32_F$ 个自由度，由下式给出

$$\left(e_\mu{}^m, \mathcal{V}_M{}^A, A_\mu^M, B_{\mu\nu\alpha}; \psi_{\mu i}, \chi_{ijk}, \chi\right), \tag{196}$$

where $i = 1, \ldots, 5$ labels the fundamental of the R -symmetry group $U(5)$, $M, A = 1, \ldots 20$ labels the20of $SU(5,1)$, and $\mathcal{V}_M{}^A$ is the $G/H = SU(5,1)/U(5)$ coset vielbein. The two-form potentials are in the adjoint representation of $SU(5,1)$. The vectors consist of 10 electric, and 10 magnetic ones, and the duality symmetry G is realized on-shell. The spinors are Weyl, the gravitini have $U(1)$ charge $+\frac{1}{2}$, the dilatinos $\chi_{ijk} = \chi_{[ijk]}$ and χ have opposite chirality, and they are in the $10 + 1$ representations of H with $U(1)$ charges $+\frac{3}{2}$ and $-\frac{5}{2}$, respectively. The two-form potential $B_{\mu\nu\alpha}$ is in the adjoint representation of $SU(5,1)$. Its properly covariantized field strengths will be related to the scalar current by a suitable duality relation.

其中 $i = 1, \ldots, 5$ 标记 R 对称群的基础表示，$U(5)$, $M, A = 1, \ldots 20$ 标记 $SU(5,1)$ 的 20 维表示，$\mathcal{V}_M{}^A$ 是 $G/H = SU(5,1)/U(5)$ 陪集标架。二形式势属于 $SU(5,1)$ 的伴随表示。矢量包含 10 个电矢量和 10 个磁矢量，对偶对称性 G 是在壳实现的。旋量为外尔旋量，引力微子带有 $U(1)$ 荷 $+\frac{1}{2}$，伸缩微子 $\chi_{ijk} = \chi_{[ijk]}$ 和 χ 手征性相反，它们分别属于 H 的 $10 + 1$ 表示，带有 $U(1)$ 荷 $+\frac{3}{2}$ 和 $-\frac{5}{2}$。二形式势 $B_{\mu\nu\alpha}$ 属于 $SU(5,1)$ 的伴随表示。其正确协变的场强将通过合适的对偶关系与标量流相联系。

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[23] The zero-center module of a superalgebra is a representation characterized by the vanishing of all super-Casimir operators.

[23] 超代数的零中心模是满足所有超卡西米尔算子都为零的表示。

The most general gauging is encoded in an embedding tensor $\theta_M{}^\alpha$, where $\alpha = 1, \ldots, 35$, and as a result of the linear constraint on it, in the product $\mathbf{20} \times \mathbf{35}$, only the $SU(5,1)$ representation $\mathbf{70} + \overline{7}0$ survives. They are described by constant tensors $\theta_{mn,p} = \theta_{[mn],p}$ satisfying $\theta_{[mn,p]} = 0$, and the complex conjugate [182]. Of course, these must also satisfy the standard quadratic constraints (310). See [182] for further details.

最一般的规范定标由嵌入张量 $\theta_M{}^\alpha$ 描述，其中 $\alpha = 1, \ldots, 35$ ，由于其满足线性约束，在乘积 $\mathbf{20} \times \mathbf{35}$ 中仅 $SU(5,1)$ 表示 $\mathbf{70} + \overline{7}0$ 保留下来。它们由满足 $\theta_{[mn,p]} = 0$ 的常数张量 $\theta_{mn,p} = \theta_{[mn],p}$ 及其复共轭 [182] 描述。当然，这些张量也必须满足标准二次约束 (310)，更多细节参见 [182]。

$N=4$ Supergravity Coupled to Vector Multiplets in 4D

$N=4$ 四维超引力与矢量多重态耦合

We combine the $N=4$ supergravity multiplet with n copies of the vector multiplet $\left(A_\mu, 6\phi;\lambda\right)$ and introduce the notation

我们将 $N=4$ 超引力多重态与 n 份矢量多重态 $\left(A_\mu, 6\phi;\lambda\right)$ 结合，并引入记号

$$\left\{e_\mu^r, \tau, \mathcal{V}_M{}^A, A_\mu^{M\alpha}, B_{\mu\nu}{}^{MN}, B_{\mu\nu}{}^{\alpha\beta}; \psi_{\mu i}, \chi_{ijk}, \lambda_{ia}\right\},\ A_\mu^M = \left\{A_\mu^0, A_\mu^\alpha\right\},$$

(197)

where $M=1,\ldots,n+6$ and $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_\alpha{}^a)$ with $m=1,\ldots,6, a=1,\ldots,n$, is a representative of the coset $SO(n,6)/SO(n)\times SO(6)$, and τ is the complex scalar that parametrizes the coset $SL(2,\mathbb{R})/SO(2)$. The index $\alpha=1,2$ labels the fundamental of $SL(2,\mathbb{R})$ and can be decomposed into the electric A_μ^{M+} and magnetic A_μ^{M-} vector fields, where $\pm$ denote the $SO(2)\subset SL(2,\mathbb{R})$ charges. The two-form potential needed by the tensor hierarchy are in the adjoint representation of $SO(n+6)\times SL(2,\mathbb{R})$. In the fermion sector, the $SU(4)\approx SO(6)$ index $i=1,\ldots,4$, and the fermions are Weyl and $\chi_{ijk}=\chi_{[ijk]}$. The $SO(2)$ charges of $\left(\psi_{\mu i}, \chi_{ijk}, \lambda_{ia}\right)$ are $\left(+\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)$. The field equations give a projected duality equations that relate the field strengths of the electric and magnetic field strengths, as well as the field strengths of the two-form fields and appropriate scalar currents, as we have already seen in many gauged supergravities surveyed so far above. Therefore the duality symmetry is on-shell.

其中带有 $m=1,\ldots,6,a=1,\ldots,n$ 的 $M=1,\ldots,n+6$ 和 $\mathcal{V}_M{}^A=(\mathcal{V}_M{}^m, \mathcal{V}_\alpha{}^a)$ 是陪集 $SO(n,6)/SO(n)\times SO(6)$ 的一个代表, 而 τ 是参数化陪集 $SL(2,\mathbb{R})/SO(2)$ 的复标量。指标 $\alpha=1,2$ 标记 $SL(2,\mathbb{R})$ 的基础表示，可分解为电矢量场 A_μ^{M+} 和磁矢量场 A_μ^{M-} ，其中 $\pm$ 表示 $SO(2)\subset SL(2,\mathbb{R})$ 荷。张量层次所需的二形式势属于 $SO(n+6)\times SL(2,\mathbb{R})$ 的伴随表示。在费米子部分, $SU(4)\approx SO(6)$ 指标为 $i=1,\ldots,4$, 且费米子是外尔费米子和 $\chi_{ijk}=\chi_{[ijk]}$。$\left(\psi_{\mu i}, \chi_{ijk}, \lambda_{ia}\right)$ 的 $SO(2)$ 荷为 $\left(+\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)$ 。如我们此前已经调研过的许多带定标超引力中所见，场方程给出了投影对偶方程，关联了电场与磁场的场强，以及二形式场的场强与对应标量流。因此对偶对称性是在线的。

The $N=4, 4D$ supergravity coupled to vector multiplets and its certain gaugings were constructed long ago [192, 193], but general gauging has been achieved in the in the embedding tensor formalism in [82], and more recently in full generality in [183], where an extensive literature on previous works can be found. See also the review [182]. A key result in the embedding tensor formalism is the characterization of the gauge group, which implies the following covariant derivative acting on representations of $SL(2)\times SO(6,n)$ [82]

$N=4, 4D$ 超引力与矢量多重态耦合及其特定定标早在文献 [192, 193] 中就已构造出来，但一般定标是在文献 [82] 中通过嵌入张量形式实现，更近文献 [183] 给出了完整的一般构造，该文还汇总了大量既往相关研究，另可见综述文献 [182]。嵌入张量形式的一个核心结果是对规范群的刻画，由此得到作用在 $SL(2)\times SO(6,n)$ 表示上的协变导数如下 [82]

$$D_\mu = \nabla_\mu - gA_\mu{}^{M\alpha}\left(f_{\alpha M}{}^{NP} - \xi_\alpha^N \delta_M^P\right)t_{NP} + gA_\mu{}^{M\alpha}\xi_M^\beta t_{\alpha\beta}, \tag{198}$$

where $f_{\alpha MNP} = f_{\alpha[MNP]}$ and ξ_M^α are the components of the constant embedding tensor that govern the gauging, and $t_{MN} = t_{[MN]}$ and $t_{\alpha\beta} = t_{(\alpha\beta)}$ are the generators of $SO(6,n)$ and $SL(2,\mathbb{R})$, respectively. This form of the embedding tensor is dictated by the linear constraint. The quadratic constraints (310) give rise to five equations which can be found in [82]. The results of [183] are the most general as they extend those of [82] as follows. Defining a symplectic frame as the choice of $(n+6)$ vector fields among the $2(n+6)$ vectors and dual vectors, a standard frame was chosen and partial results for the Lagrangian and supertransformations were given in [82]. On the other hand, arbitrary symplectic frames were considered, and the full Lagrangian and supertransformations were provided in [183]. For further details such as particular solutions of the constraints, and possible connections between the resulting gauge supergravities and type IIB flux compactifications, see [182,183], and the references therein. Relation between gauge $N=4, 4D$ supergravity coupled to vector multiplets and the double field theory formulation of $N=1, 10D$ heterotic supergravity also offers an approach to its embedding to string theory [194, 195].

其中 $f_{\alpha MNP} = f_{\alpha[MNP]}$ 和 ξ_M^α 是支配规范群的恒定嵌入张量分量，$t_{MN} = t_{[MN]}$ 和 $t_{\alpha\beta} = t_{(\alpha\beta)}$ 分别是 $SO(6,n)$ 和 $SL(2,\mathbb{R})$ 的生成元。该形式的嵌入张量由线性约束决定。二次约束 (310) 给出了五个方程，可见文献 [82]。文献 [183] 的结果最具一般性，它对文献 [82] 的结果拓展如下: 将辛框架定义为在 $2(n+6)$ 向量和对偶向量中选取 $(n+6)$ 个向量场，文献 [82] 选取了标准框架，给出了拉格朗日量和超变换的部分结果；而文献 [183] 考虑了任意辛框架，给出了完整的拉格朗日量和超变换。关于约束特解、所得规范超引力与 IIB 型通量紧致化之间可能的联系等更多细节，参见文献 [182,183] 及其引文。耦合了向量多重态的规范 $N=4, 4D$ 超引力与 $N=1, 10D$ 杂化超引力的双重场论表述之间的关系，也为将其嵌入弦论提供了一种方法 [194, 195]。

$N=3$ Supergravity Coupled to Vector Multiplets in 4D

$N=3$ 四维下耦合矢量多重态的超引力

The complete $N=3$ supergravity coupled to vector multiplets was constructed in [196], using the group manifold approach. Combining the supergravity multiplet with n vector multiplet $(A_\mu, 6\phi; \lambda_i, \lambda)$, where $i = 1,2,3$, gives the field content

完整的耦合矢量多重态的 $N=3$ 超引力由文献 [196] 利用群流形方法构造得到。将超引力多重态与 n 个矢量多重态 $(A_\mu, 6\phi; \lambda_i, \lambda)$ 结合，其中 $i=1,2,3$， 得到的场内容为

$$\left\{e_\mu^r, V_M{}^A, A_\mu^M, B_{\mu\nu\alpha}; \psi_{\mu i}, \chi, \lambda_{ia}, \lambda\right\}, \tag{199}$$

where $M = 1,\dots,n+3$ and $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a)$ with $m = 1,\dots,3, a = 1,\dots,n$, is a representative of the coset $SU(n,3)/SU(n)\times SU(3)$. The two-form potentials are in the adjoint representation of $SU(n,3)$. In the fermionic sector, the $SU(3)$ index $i = 1,\dots,3$, and the fermions are Weyl. The $U(1)_R \subset U(3)_R$ charges of the fermions $(\psi_{\mu i}, \chi, \lambda_{ia}, \lambda)$ are given by $(1/2, 3/2, (n+6)/2n, 3(n+2)/2n)$, respectively. Subgroups of $SO(n,3) \subset SU(n,3)$ of dimensions $n+3$ were gauged in [196]. As observed in [197], there is a similarity with the allowed gauged groups in half-maximal $7D$ supergravity coupled to n vector multiplet. As such, the gauge groups listed in (66) were found in the present case. More general gauging based on the embedding tensor formalism is described in [182], where it is shown that the embedding tensor is restricted by the linear constraint to transform as an irreducible $SU(n,3)$ tensor $\theta_{MN}{}^P = \theta_{[MN]}{}^P$ and its conjugate, $\theta^{MN}{}_P$. The quadratic constraints (310) need to be imposed as well. A systematic analysis of these constraints and a classification of all possible gaugings remain to be carried out.

其中 $M = 1, \ldots, n+3$ 和 $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a)$ (满足 $m = 1, \ldots, 3, a = 1, \ldots, n$) 是陪集 $SU(n,3)/SU(n) \times SU(3)$ 的一个代表元。二形式势位于 $SU(n,3)$ 的伴随表示下。在费米子部分, $SU(3)$ 指标 $i = 1, \ldots, 3$ ，且所有费米子都是外尔费米子。费米子 $(\psi_{\mu i}, \chi, \lambda_{ia}, \lambda)$ 的 $U(1)_R \subset U(3)_R$ 荷分别由 $(1/2, 3/2, (n+6)/2n, 3(n+2)/2n)$ 给出。文献 [196] 对维数为 $n+3$ 的 $SO(n,3) \subset SU(n,3)$ 子群做了定规范。正如文献 [197] 指出的，它和半极大 $7D$ 超引力耦合 n 个矢量多重态中允许的定规范群存在相似性，因此式 (66) 列出的规范群也适用于本文的情况。文献 [182] 描述了基于嵌入张量形式论的更一般定规范，其中表明，线性约束要求嵌入张量只能变换为不可约 $SU(n,3)$ 张量 $\theta_{MN}{}^P = \theta_{[MN]}{}^P$ 及其共轭 $\theta^{MN}{}_P$ 。同时还需要满足二次约束 (310)。对这些约束的系统分析以及所有可能定规范的分类仍有待完成。

$N = 2$ Supergravity Coupled to Scalar and Vector Multiplets in 4D

$N = 2$ 四维下耦合标量多重态与矢量多重态的超引力

Combining the $N = 2, 4D$ supergravity multiplet, which contains a single vector, with n_V vector multiplets each containing a complex scalar z , and n_T hypermultiplets, upon introducing the magnetic vector potentials and two-form fields that are dual to the scalar currents, we get the field content

将包含单个矢量的 $N = 2, 4D$ 超引力多重态，与每个都包含一个复标量 z 的 n_V 个矢量多重态，以及 n_T 个超多重态结合，引入对偶于标流的磁矢量势和二形式场后，我们得到场内容

$$\left\{e_\mu^r, A_\mu^M, z^\alpha, B_{\mu\nu\alpha}; \psi_\mu^i, \lambda^{\alpha i}\right\}, \left\{\phi^X, \psi^a\right\},$$

where

其中

$$i = 1, 2, \alpha = 1, \ldots, n_V, X = 1, \ldots, 4n_H, a = 1, \ldots, 2n_H$$

$$A_\mu^M = \left(A_\mu^\Lambda, A_{\mu\Lambda}\right), \ \Lambda = 0, 1 \ldots, n_V. \tag{200}$$

The n_V complex scalars, z^α , parametrize a special Kähler (SK) manifold $\mathcal{M}_{SK}$, described below, and ϕ^X parametrize a quaternionic Kähler manifold $\mathcal{M}_{QK}$, described in section " $(1,0)$ Supergravity Coupled to Vector, Tensor, and Hyper Multiplets in 6D", as required by supersymmetry. As a special case of SK geometries, the so-called VSK geometries arise from the dimensional reduction of the VSR geometries present in $5D$ supergravities via the r -map, as mentioned briefly at the end of section "D=5". See Table 5 in Appendix C for a list of supergravities in which SK and VSK geometries arise. These geometries and the relationships between them are treated in great detail for example, in [118, 182].

n_V 个复标量 z^α 参数化了一个将在下文介绍的特殊凯勒 (SK) 流形 $\mathcal{M}_{SK}$ ，而根据超对称要求，ϕ^X 参数化了一个四元数凯勒流形 $\mathcal{M}_{QK}$ ，该流形在章节“(1,0) 六维超引力与矢量、张量和超多重子的耦合”中有所介绍。作为特殊凯勒几何的一个特例，所谓 VSK 几何，正如“五维”章节末尾简要提及的，是通过 r 映射对 5D 超引力中存在的 VSR 几何做维约化得到的。附录 C 的表 5 列出了存在 SK 和 VSK 几何的超引力。例如，文献 [118, 182] 已经对这些几何及其相互关系进行了非常详尽的讨论。

A good source for original papers on $N = 2, 4D$ supergravities coupled to vector and scalar multiplets is [198]. The general electric gaugings of $N = 2, 4D$ supergravity coupled to vector multiplets are given by using the group manifold approach in [149, 199], and the duality covariant gaugings in [200] by means of superconformal tensor calculus. The duality covariant couplings were also obtained in [201] by a direct computation. See also [118]. Here we shall summarize briefly the detailed account given in [182], which primarily follows [201].

关于 $N = 2, 4D$ 超引力耦合矢量和标量多重态的原始文献，[198] 是很好的资料来源。$N = 2, 4D$ 超引力耦合矢量多重态的一般电规范，在 [149, 199] 中利用群流形方法给出，对偶协变规范则在 [200] 中借助超共形张量微积分给出。对偶协变耦合也在 [201] 中通过直接计算得到。另参见文献 [118]。本文将简要概述 [182] 中的详细说明，该内容主要沿用 [201] 的结论。

Some of the key ingredients in the description of SK manifolds are the symplectic vectors and Kähler potentials which take the form

描述特殊凯勒流形的一些关键要素是辛向量和具有如下形式的凯勒势

$$V^M = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \; v^M(z) = \begin{pmatrix} L^\Lambda(z) \\ M_\Lambda(z) \end{pmatrix}, \; v^M(z) = e^{-\mathcal{K}/2} V^M$$

$$\mathcal{K}(z,\bar{z}) = -\log\left[i\left(\bar{v}^T \Omega v\right)\right], \; \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}. \tag{201}$$

It is also useful to define the Kähler covariant derivatives

定义凯勒协变导数也十分有用

$$U_\alpha^M \equiv \begin{pmatrix} f_\alpha{}^\Lambda \\ h_{\alpha\Lambda} \end{pmatrix} = \nabla_\alpha V^M = \partial_\alpha V^M + \frac{1}{2}\left(\partial_\alpha \mathcal{K}\right) V^M. \tag{202}$$

For SK manifolds the following conditions must be satisfied

特殊凯勒流形必须满足以下条件

$$\nabla_\alpha V^M = 0, \; \nabla_\alpha U_\beta^M = i C_{\alpha\beta\gamma} g^{\gamma\bar{\gamma}} \bar{U}_{\bar{\gamma}}^M,$$

$$\nabla_\alpha \bar{U}_{\bar{\beta}} = g_{\alpha\bar{\beta}} \bar{V}^M, \; V^M \Omega_{MN} U_\alpha^N = 0, \tag{203}$$

where $C_{\alpha\beta\gamma}$ is a total symmetric tensor. To describe the Yang-Mills kinetic terms, it proves useful to define [118]

其中 $C_{\alpha\beta\gamma}$ 是全对称张量。为描述杨-米尔斯动能项，按 [118] 定义如下量是有用的

$$\mathcal{N}_{IJ} = \left(F_{\Lambda}\nabla_{\overline{\alpha}}\bar{F}_{\Lambda}\right)\left(X^{\Sigma}\nabla_{\overline{\alpha}}\bar{X}^{\Sigma}\right)^{-1}. \tag{204}$$

Next, consider the gauging of a subgroup of the isometry group $G_{SK} \times G_{QK}$ of the scalar manifold $\mathcal{M}_{SK} \times \mathcal{M}_{QK}$ with generators (t_u, t_m) , respectively. Thus, $u = 1, \ldots, \dim G_{SK}$ and $m = 1, \ldots, \dim G_{QK}$. The gauge generators X_M are expressed as

接下来，考虑对标量流形 $\mathcal{M}_{SK} \times \mathcal{M}_{QK}$ 等距群 $G_{SK} \times G_{QK}$ 的子群做规范，其生成元为 (t_u, t_m) ，因此有 $u = 1, \ldots, \dim G_{SK}$ 和 $m = 1, \ldots, \dim G_{QK}$ 。规范生成元 X_M 可表示为

$$X_M = \theta_M{}^u t_u + \theta_M{}^m t_m,\ X_{MN}{}^P = \theta_M{}^u (t_u)_N{}^P. \tag{205}$$

The resulting linear and quadratic constraints on the embedding tensors are [182]

嵌入张量满足的线性和二次约束为 [182]

$$X_{(MNP)} = 0,\ \theta_M{}^u \theta_N{}^v f_{uv}{}^w + X_{MN}{}^P \theta_P{}^w = 0, \tag{206}$$

$$\theta_M{}^m \theta_N{}^n f_{mn}{}^p + X_{MN}{}^P \theta_P{}^p = 0,\ \theta^{MA}\theta_M{}^B = 0,\ A = (u, m).$$

The moment maps are given by [182]

矩映射由 [182] 给出

$$\mathcal{P}_m^x = \frac{1}{2\lambda n_H} J^x{}_Y{}^X D_X k_m{}^Y,\ x = 1, 2, 3, \tag{207}$$

where $J^x{}_Y{}^X$ are the quaternionic structures, $k_m{}^Y$ are the Killing vectors generating the algebra of the generators t^m on $\mathcal{M}_{QK}$, which has negative constant scalar curvature $R = 8\lambda n_H (n_H + 2)$ that defines the constant λ .

其中 $J^x{}_Y{}^X$ 是四元数结构，$k_m{}^Y$ 是生成元 t^m 在 $\mathcal{M}_{QK}$ 上生成代数的基灵矢量，$\mathcal{M}_{QK}$ 具有负常数标量曲率 $R = 8\lambda n_H (n_H + 2)$ ，该曲率定义了常数 λ 。

With the above definitions at hand, the bosonic part of the Lagrangian is given by [118]

有了上述定义，拉格朗日量的玻色子部分由 [118] 给出

$$e^{-1}\mathcal{L} = \frac{1}{2}R - g_{\alpha\overline{\beta}} D_\mu z^\alpha D^\mu \bar{z}^{\overline{\beta}} - \frac{1}{2} g_{XY} D_\mu \phi^X D^\mu \phi^Y$$

$$+\frac{i}{4}\left(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\mu\nu}\Sigma-\text{ h.c. }\right)+\mathcal{L}_{\text{ top}}-V \tag{208}$$

where

其中

$$D_\mu z^\alpha=\partial_\mu z^\alpha-A^M_\mu k_M{}^\alpha,\ D_\mu\phi^X=\partial_\mu\phi^X-A^M_\mu k_M{}^X,$$

$$k_M{}^\alpha=\theta_M{}^u k_u{}^\alpha,\ k_M{}^X=\theta_M{}^m k_m{}^X, \tag{209}$$

and $k_u{}^\alpha$ are the Killing vectors generating the algebra of the generators t^u on $\mathcal{M}_{SK}$. All Killing vectors preserve the complex and quaternionic structures. The field strength $\mathcal{F}^\Lambda_{\mu\nu}$ is defined as

且 $k_u{}^\alpha$ 是生成元 t^u 在 $\mathcal{M}_{SK}$ 上生成代数的基灵矢量。所有基灵矢量都保持复结构和四元数结构。场强 $\mathcal{F}^\Lambda_{\mu\nu}$ 定义为

$$\mathcal{F}^\Lambda_{\mu\nu}=2\partial_{[\mu}A^\Lambda_{\nu]}+gX_{[NP]}{}^\Lambda A^N_\mu A^P_\nu+\frac{1}{2}g\theta^{\Lambda A}B_{\mu\nu A}, \tag{210}$$

where we recall that $A=(u,m)$. The potential which takes the form

其中我们已知 $A=(u,m)$ 。势的形式为

$$V=g^2\left[\left(k_M{}^\alpha k_N{}^{\bar\beta}g_{\alpha\bar\beta}+4g_{XY}k_M{}^X k_M{}^Y\right)\bar V^M V^N+\left(U^{MN}-3V^M\bar V^N\right)\mathcal{P}^x_N\mathcal{P}^x_N\right],$$

(211)

where $U_{MN}\ :=\ g^{\alpha\bar\beta}U^M_\alpha\bar U^N_{\bar\beta}$. The Lagrangian $\mathcal{L}_{\text{ top}}$ has the same for as in (192), and the resulting duality equation which relates the three-form field strength (which can be deduced from the Bianchi identity for the two-form field strength defined above) to the dual of the scalar current has a form similar to (193).

其中 $U_{MN}\ :=\ g^{\alpha\bar\beta}U^M_\alpha\bar U^N_{\bar\beta}$ 。拉格朗日量 $\mathcal{L}_{\text{ top}}$ 与 (192) 式形式相同，得到的对偶方程将三形式场强 (可由上述定义的二形式场强的比安基恒等式推导得到) 与标量流的对偶联系起来，其形式与 (193) 式类似。

Finally, supertransformations of the fermions are given by [24]

最后，费米子的超变换由 [24] 给出

$$\delta\psi_{\mu i}=D_\mu\varepsilon_i+\frac{1}{2}\varepsilon_{ij}H^-_{\mu\nu}\varepsilon^j+ig\gamma_\mu S_{ij}\varepsilon^j,$$

$$\delta\lambda^{\alpha i}=i\gamma^\mu D_\mu z^\alpha\gamma^\mu\varepsilon^i+\frac{i}{4}H^{-\alpha}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_j\varepsilon^{ij}+gW^{\alpha ij}\varepsilon_j,$$

$$\delta\psi_a=i\mathcal{V}_X{}^{bj}D_\mu\phi^X\gamma^\mu\varepsilon_{ij}\Omega_{ab}+gN_a{}^i\varepsilon_i, \tag{212}$$

where $\mathcal{V}_X{}^{ai}$ are the vielbeins on $\mathcal{M}_{QK}$, and

其中 $\mathcal{V}_X{}^{ai}$ 是 $\mathcal{M}_{QK}$ 上的标架，且

$$D_\mu \varepsilon_i = \left(\nabla_\mu - \frac{i}{2} Q_\mu\right) \varepsilon_i - Q_{\mu j}{}^i \varepsilon^j,$$

$$Q_\mu = \left(\frac{i}{2} \partial_\alpha K \partial_\mu z^\alpha - \text{ h.c. }\right) - A_\mu^I P_I^0, \; Q_\mu{}^{ij} = -\omega_X{}^{ij} \partial_\mu \phi^X - \frac{1}{2} A_\mu^M \mathcal{P}_M{}^{ij},$$

$$H_{\mu\nu}^- = 2iL^\Lambda \left(\operatorname{Im} \mathcal{N}_{\Lambda\Sigma}\right) \mathcal{H}_{\mu\nu}^{\Sigma -}, \; H_{\mu\nu}^{-\alpha} = 2ig^{\alpha\bar{\beta}} \bar{f}_{\bar{\beta}}^{\Lambda} \left(\operatorname{Im} \mathcal{N}_{\Lambda\Sigma}\right) H_{\mu\nu}^{\Sigma -}, \tag{213}$$

with $\omega_X{}^{ij}$ representing the composite $Sp(1)$ connection on $\mathcal{M}_{QK}$, and $\mathcal{P}_M{}^x := \Theta_M{}^m \mathcal{P}_m{}^x$. The shift functions are given by

其中 $\omega_X{}^{ij}$ 代表 $\mathcal{M}_{QK}$ 上的复合 $Sp(1)$ 联络，且 $\mathcal{P}_M{}^x := \Theta_M{}^m \mathcal{P}_m{}^x$ 。平移函数由下式给出

$$S_{ij} = \frac{1}{2} i (\sigma^x)_i{}^k \varepsilon_{jk} \mathcal{P}_M^x V^M, \; N_a{}^i = -2 \mathcal{V}_X{}^i{}_a k_M{}^X \bar{V},$$

$$W^{\alpha ij} = -\varepsilon^{ij} k_M{}^\alpha \bar{V}^M + i (\sigma^x)_k{}^j \varepsilon^{ki} \mathcal{P}_M{}^x g^{\alpha\bar{\beta}} \bar{U}_{\bar{\beta}}^M. \tag{214}$$

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[24] In this section we use the conventions of [182], where $X^M = -\Omega^{MN} X_N, X_M = X^N \Omega_{NM}$. Furthermore, the position of the index i on a spinor is associated with its chirality, such that $\psi_{\mu i}, \lambda_i^\alpha$ have positive chirality, while $\psi_\mu^i, \lambda^{i\alpha}$ have negative chirality. The hyperino ψ^a has positive chirality.

[24] 本节我们采用文献 [182] 的约定，其中 $X^M = -\Omega^{MN} X_N, X_M = X^N \Omega_{NM}$ 。此外，旋量上指标 i 的位置与手征性相关，即 $\psi_{\mu i}, \lambda_i^\alpha$ 为正手征，$\psi_\mu^i, \lambda^{i\alpha}$ 为负手征。希格蒂诺 ψ^a 为正手征。

For a wealth of information on the subject we have barely touched, see, for example, the book [118], and the references therein.

对于本节我们 barely 触及的该主题的更多内容，可参见例如专著 [118] 及其中的参考文献。

$N = 1$ Supergravity Coupled to Scalar and Vector Multiplets in 4D

$N = 1$ 四维下耦合标量多重态与矢量多重态的超引力

It is natural that the general matter coupled $N = 1$ supergavity is the most studied one for many years. The general couplings were given long ago in [202]. Its structure has been thoroughly treated in the excellent book by Freedman and van Proeyen [181], where some applications are discussed as well. See also the reviews [203,204]. For the sake of completeness, we shall give the bosonic part of the matter coupled $N = 1$ supergravity action and the supersymmetry transformations here.

多年来，普遍耦合物质的 $N = 1$ 超引力一直是研究最多的课题，这十分自然。普遍耦合早在文献 [202] 中就已给出。弗里德曼和范普罗恩在其优秀著作 [181] 中对其结构做了详尽论述，还讨论了若干应用，也可参见综述文献 [203,204]。为完备起见，我们在此给出耦合物质的 $N = 1$ 超引力作用量的玻色子部分以及超对称变换。

The supergravity, chiral, and Yang-Mills multiplets have the fields $\left(e_\mu^a, \psi_\mu\right)$, $\left(z^\alpha, \chi_L^\alpha\right)$, and $\left(A_\mu^A, \lambda^A\right)$, with $\alpha = 1, \ldots, n$ and $A = 1, \ldots, \dim G$. As is well known, the complex scalar fields of the n chiral multiplets, denoted by z^α here, parametrize a Hodge-Kähler manifold $\mathcal{M}$. We shall assume that $\mathcal{M}$ has an isometry group G which is fully gauged by the introduction of the Yang-Mills multiplet. [25] Prior to giving the bosonic part of the full action, it is useful to specify their building blocks.

超引力多重态、手征多重态与杨-米尔斯多重态包含的场为 $\left(e_\mu^a, \psi_\mu\right)$、$\left(z^\alpha, \chi_L^\alpha\right)$ 和 $\left(A_\mu^A, \lambda^A\right)$，带有 $\alpha = 1, \ldots, n$ 和 $A = 1, \ldots, \dim G$。众所周知，n 个手征多重态的复标量场 (本文记为 z^α) 参数化了一个霍奇-凯勒流形 $\mathcal{M}$。我们假设 $\mathcal{M}$ 具有等距群 G，通过引入杨-米尔斯多重态可以对该群完全定规范。[25] 在给出完整作用量的玻色子部分之前，先明确其构造模块会更清晰。

The key building blocks are the Kähler potential $\mathcal{K}(z, \bar{z})$, holomorphic super-potential $W(z)$, holomorphic gauge function $f_{AB}(z) = f_{(AB)}(z)$, and holomorphic Killing vectors $k_A{}^\alpha(z)$. The metric $g_{\alpha\bar{\beta}}$ is invariant under the Kähler transformations $\mathcal{K}(z, \bar{z}) \rightarrow \mathcal{K}(z, \bar{z}) + F(z) + \bar{F}(\bar{z})$, but under the gauge transformation that acts as $\delta z^\alpha = \theta^A(x) k_A{}^\alpha(z)$, where $\theta(x)^A$ is the gauge parameter, the Kähler potential K is invariant up to a gauge transformation, viz.

核心构造模块包括凯勒势 $\mathcal{K}(z, \bar{z})$、全纯超势 $W(z)$、全纯规范函数 $f_{AB}(z) = f_{(AB)}(z)$ 和全经基灵矢量 $k_A{}^\alpha(z)$。度规 $g_{\alpha\bar{\beta}}$ 在凯勒变换 $\mathcal{K}(z, \bar{z}) \rightarrow \mathcal{K}(z, \bar{z}) + F(z) + \bar{F}(\bar{z})$ 下不变，而在作用为 $\delta z^\alpha = \theta^A(x) k_A{}^\alpha(z)$ 的规范变换下 (其中 $\theta(x)^A$ 是规范参数)，凯勒势 K 至多差一个规范变换保持不变，即:

$$\delta_\theta \mathcal{K} = \theta^A k_A{}^\alpha \partial_\alpha \mathcal{K} + \text{ h.c. } = \theta^A r_A(z) + \text{ h.c. }, \tag{215}$$

where $r_A(z)$ is defined by this equation. Invariance of the action under the Kähler transformation also requires that $W \rightarrow e^{-F} W$. Further conditions that need to be satisfied for gauge and Kähler invariance are [181, 203]

式中 $r_A(z)$ 由该方程定义。作用量在凯勒变换下的不变性还要求 $W \rightarrow e^{-F} W$。规范不变性与凯勒不变性所需满足的进一步条件为 [181, 203]

$$k_A{}^\alpha g_{\alpha\bar{\beta}} k_B{}^{\bar{\beta}} - A \leftrightarrow B = i f_{AB}{}^C \mathcal{P}_C, \; k_A{}^\alpha \nabla_\alpha W = i \mathcal{P}_A,$$

$$k_C{}^\alpha \partial_\alpha f_{AB}(z) = 2 f_{C(A}{}^D f_{B)D}(z) + i C_{AB,C}, \tag{216}$$

where $C_{AB,C} = C_{(AB),C}$ are real constants, and the real scalar moment map

式中 $C_{AB,C}=C_{(AB),C}$ 为实常数，实标量矩映射

$$\mathcal{P}_A(z,\bar{z})=i\left[k_A{}^{\alpha}(z)\,\partial_\alpha\mathcal{K}(z,\bar{z})-r_A(z)\right], \tag{217}$$

which is defined up to arbitrary constant shifts, is related to the Fayet-Iliopoulos D -terms in the presence of abelian gauge group factors. It is also useful to specify the gauge covariant derivative $D_\mu z^\alpha$, and the Kähler covariant derivative $\nabla_\alpha W$,

它可相差任意常数平移，当存在阿贝尔规范群因子时，它与法耶特-伊里奥普洛斯 D 项相关。明确给出规范协变导数 $D_\mu z^\alpha$ 和凯勒协变导数 $\nabla_\alpha W$ 也便于后续讨论，

$$D_\mu z^\alpha=\partial_\mu-A_\mu^A k_A{}^{\alpha},\ \nabla_\alpha W=\partial_\alpha W+(\partial_\alpha K)\,W. \tag{218}$$

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[25] Gauging a subgroup of G is straightforward, with minimal adjustment of notation.

[25] 对 G 的子群定规范十分直接，仅需对记号做最小调整。

With the above ingredients at hand, the bosonic part of the Lagrangian is given by [202, 205, 206]

利用上述要素，拉格朗日量的玻色子部分可写为 [202, 205, 206]

$$e^{-1}\mathcal{L}=\frac{1}{2}R-g_{\alpha\bar{\beta}}D_\mu z^\alpha D^\mu z^{\bar{\beta}}-\frac{1}{4}(\mathrm{Re}\,f_{AB})\,F_{\mu\upsilon}^A F^{\mu\upsilon B}-i\frac{1}{4}(\mathrm{Im}\,f_{AB})\,F_{\mu\upsilon}^A\tilde{F}^{\mu\upsilon B}$$

$$-e^K\left(\nabla_\alpha W g^{\alpha\bar{\beta}}\overline{\nabla}_{\bar{\beta}}\bar{W}+3W\bar{W}-\frac{1}{2}(\mathrm{Re}\,f)^{-1AB}\mathcal{P}_A\mathcal{P}_B\right)+e^{-1}\mathcal{L}_{CS}$$

(219)

where $F_{\mu\upsilon}^A=2\partial_{[\mu}A_{\upsilon]}^A+gf_{BC}{}^{A}A_\mu^B A_\upsilon^C$ and the Chern-Simons Lagrangian is given by [206]

其中 $F_{\mu\upsilon}^A=2\partial_{[\mu}A_{\upsilon]}^A+gf_{BC}{}^{A}A_\mu^B A_\upsilon^C$ ，陈-西蒙斯拉格朗日量由下式给出 [206]

$$\mathcal{L}_{CS}=\frac{1}{2}\left(C_{AB,C}-C_{(AB,C)}\right)\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{3}A_\mu^C A_\nu^A F_{\rho\sigma}^B+\frac{1}{4}f_{DE}{}^{A}A_\mu^D A_\nu^E A_\rho^C A_\sigma^B\right).$$

(220)

If the theory has gauge anomalies, they will be encoded in a totally symmetric constant tensor d_{ABC} , and they can be removed by shifting the C -dependent factor in $\mathcal{L}_{CS}$ by d_{ABC} [205, 206] .

若该理论存在规范反常，它们会被编码进全对称常数张量 d_{ABC} 中，只需将 $\mathcal{L}_{CS}$ 中依赖 C 的因子平移 d_{ABC} [205, 206] 即可消除这些反常。

The supertransformations of the fermions are

费米子的超变换为

$$\delta\psi_{\mu L} = D_\mu \varepsilon_L + \frac{1}{2} e^{\mathcal{K}/2} \gamma_\mu \varepsilon_R$$

$$\delta\chi_L^\alpha = \frac{1}{\sqrt{2}} \gamma^\mu D_\mu z^\alpha \varepsilon_R - \frac{1}{\sqrt{2}} e^{K/2} g^{\alpha\bar{\beta}} \overline{\nabla}_{\bar{\beta}} \bar{W} \varepsilon_L$$

$$\delta\lambda^A = \frac{1}{4} \gamma^{\mu\nu} F^A_{\mu\nu} \varepsilon + \frac{1}{4} i (\mathrm{Re}\, f)^{-1AB} P_B \varepsilon, \tag{221}$$

where $D_\mu \varepsilon_L = \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} - \frac{3}{2} i Q_\mu\right) \varepsilon_L$ with

其中 $D_\mu \varepsilon_L = \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} - \frac{3}{2} i Q_\mu\right) \varepsilon_L$ 满足

$$Q_\mu = \frac{i}{6} \left(\partial_\mu z^\alpha \partial_\alpha \mathcal{K} - \text{ h.c. }\right) - \frac{1}{3} A_\mu^A \mathcal{P}_A. \tag{222}$$

Invariance of the action under the Kähler transformation requires, in addition to $W \to e^{-F} W$, that

作用量在凯勒变换下的不变性除要求 $W \to e^{-F} W$ 外，还要求

$$\psi_{\mu L} \to e^{-iF/2} \psi_{\mu L}, \lambda_L^A \to e^{-iF/2} \lambda_L^A, \chi_L^\alpha \to e^{iF/2} \chi_L^\alpha. \tag{223}$$

While globally $N = 1$ supersymmetric sigma models exist for all Kähler manifolds, only a subclass known as Hodge manifolds can be coupled in a globally well defined manner to supergravity. A Hodge manifold is a Kähler manifold on which it is possible to define a complex line bundle whose first Chern class is proportional to the Kähler form $J := (i/2)\left(\partial^2 K / \partial z^\alpha \partial z^{\bar{\beta}}\right) dz^\alpha \wedge dz^{\bar{\beta}}$. For topologically non-trivial Hodge manifolds this leads to the quantization of Newton's constant in terms of the scalar self-coupling. [26] For more details, see [181,203]. While scalar manifolds for $N > 1$ are always non-compact, in the case of $N = 1$ compact scalar manifolds are possible [181]. Finally, we note that under certain circumstances, the Kähler symmetry may develop anomalies at the quantum level. For a discussion of these anomalies, see [208] and the references therein.

尽管整体 $N = 1$ 超对称西格玛模型对所有凯勒流形都存在，但只有名为霍奇流形的子类能以整体良定义的方式耦合到超引力。霍奇流形是一类凯勒流形，其上可以定义复线丛，且该线丛的第一陈类正比于凯勒形式 $J := (i/2)\left(\partial^2 K / \partial z^\alpha \partial z^{\bar{\beta}}\right) dz^\alpha \wedge dz^{\bar{\beta}}$ 。对于拓扑非平凡的霍奇流形，这会导致牛顿常数按标量自耦合量子化。[26] 更多细节参见 [181,203]。尽管 $N > 1$ 的标量流形始终是非紧致的，但在 $N = 1$ 的情况下可以存在紧致标量流形 [181]。最后我们指出，在特定条件下，凯勒对称性可能在量子层面出现反常，关于这些反常的讨论参见 [208] 及其中的参考文献。

$\mathbf{D = 3}$

The on-shell N -extended supergravity multiplet consists of the vielbein and gravi-tini, $\left(e_\mu{}^a, \psi_\mu^I\right)$, where $I = 1, \ldots, N$. The two-derivative supergravity theory for this multiplet is topological. An $N = 8$ non-topological theory is obtained by coupling to n scalar multiplets whose scalars parametrize the coset $SO(n, 8) / (SO(n) \times$

$SO(8))$, and an $N=16$ theory is obtained by coupling to 128 scalar multiplets whose scalars parametrize the coset $E_{8(8)}/SO(16)$, as was shown in [209]. [27] These results were generalized to obtain other general ungauged $N<16$ supergravities coupled to nonlinear sigma models in [211] where the geometries of the scalar manifolds were identified. A subset of the scalars can be dualized to vector fields, which in turn can be used to gauge certain isometries of the scalar manifolds. In view of the duality relation, it was shown in [212] that there are two types of gaugings, known as Chern-Simons (CS) type and Yang-Mills type gaugings, depending on whether the scalars of the vectors carry the propagating degrees of freedom. Below we shall give a summary of the CS gaugings described in detail in [152]. Those which admit an AdS vacuum solution with $\mathrm{OSp}(2\mid p)\otimes\mathrm{OSp}(2\mid q)$ supersymmetry, upon truncation down to the topological theory, may be referred to as $N=(p,q)$ supergravity, constructed long ago in [213]. The topological theory is based on gauging the $N=(p,q)$ super AdS algebra and it exists by itself for any N .

壳上 N 扩展超引力多重态由 vielbein 和引力微子 $\left(e_\mu{}^a,\psi_\mu^I\right)$ 组成，其中 $I=1,\dots,N$ 。该多重态的二微商超引力理论是拓扑的。通过耦合 n 个标量多重态可得到一个 $N=8$ 非拓扑理论，这些标量多重态的标量参数化陪集 $SO(n,8)/(SO(n)\times\ SO(8))$ ；而通过耦合 128 个标量多重态可得到一个 $N=16$ 理论，这些标量多重态的标量参数化陪集 $E_{8(8)}/SO(16)$ ，这已在文献 [209] 中证明。[27] 这些结果被推广，在文献 [211] 中得到了其他耦合非线性 sigma 模型的一般非规范 $N<16$ 超引力，并确定了标量流形的几何。一部分标量可以对偶化为矢量场，进而可用于规范标量流形的某些等距。利用对偶关系，文献 [212] 证明存在两类规范，分别称为陈-西蒙斯 (CS) 型和杨-米尔斯型规范，区分依据是矢量对应的标量是否承载传播自由度。下文我们将总结文献 [152] 中详细描述的 CS 规范。那些截断到拓扑理论后仍允许存在带 $\mathrm{OSp}(2\mid p)\otimes\mathrm{OSp}(2\mid q)$ 超对称的 AdS 真空解的理论，可称为 $N=(p,q)$ 超引力，早在文献 [213] 中就已构造完成。该拓扑理论基于规范 $N=(p,q)$ 超 AdS 代数，且对任意 N 都自洽存在。

All Gauged On-Shell Supergravities in 3D

三维下所有规范离壳超引力

Supergravity in $3D$ coupled to d real scalar fields has the field content [152, 211]

$3D$ 维超引力耦合到 d 个实标量场，其场内容为 [152, 211]

$$\left\{e_\mu{}^a,\psi_\mu^i,\phi^\alpha,\chi^{\alpha i}\right\},\tag{224}$$

where $i=1,\dots,N$ and $\alpha=1,\dots,d$. The fermions are Majorana, and the scalars parametrize a d dimensional target space. We shall primarily follow [152] in our brief review here. Supersymmetry leads to stringent conditions on the target space, which can only be satisfied when the number of supersymmetries is restricted to $N\le 16$, and it has to admit $N(N-1)/2$ complex structures $\left(f^{ij}\right)_\alpha{}^\beta$. For $N>2$ the target space has to be Einstein with $R_{\alpha\beta}=\left((N-2)+\frac{d}{8}\right)g_{\alpha\beta}>0$. In particular, for $N=3$ the target space has to be quaternionic Kahler, while for $N=4$ it has to be a direct product of two quaternionic spaces of dimension $d_\pm$. For $N>4$, the target space has to be homogeneous. There exists no theory with $N=7$ supersymmetry and beyond $N=8$ there are only four possible theories. They are $N=9,10,12,16$ supersymmetric, and their target spaces, which are unique, are respectively the symmetric spaces,

其中 $i=1,\ldots,N$ 和 $\alpha=1,\ldots,d$ 。费米子为马约拉纳费米子，标量参数化一个 d 维目标空间。本文的简要综述主要遵循文献 [152]。超对称对目标空间施加了严格条件，这些条件仅当超对称数限制为 $N\leq 16$ 时才能满足，且目标空间必须允许 $N(N-1)/2$ 个复结构 $\left(f^{ij}\right)_{\alpha}{}^{\beta}$ 。对于 $N>2$ ，目标空间必须是带有 $R_{\alpha\beta}=\left((N-2)+\frac{d}{8}\right)g_{\alpha\beta}>0$ 的爱因斯坦空间。特别地，对于 $N=3$ ，目标空间必须是四元数凯勒空间，而对于 $N=4$ ，它必须是两个维数为 $d_{\pm}$ 的四元数空间的直积。对于 $N>=4$, 目标空间必须是齐次的。不存在具有 $N=7$ 超对称的理论，且在 $N=8$ 之外仅存在四种可能的理论。它们分别是 $N=9,10,12,16$ 超对称，其唯一的目标空间依次为如下对称空间:

$$\frac{F_{4(-20)}}{SO(9)},\ \frac{E_{6(-14)}}{SO(10)\times SO(2)},\ \frac{E_{7(7)}}{SO(12)\times SO(3)},\ \frac{E_{8(8)}}{SO(16)}. \tag{225}$$

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[26] For the noncompact case with nontrivial topology, the story is more complicated; see [207].

[26] 对于具有非平凡拓扑的非紧致情况，情况更复杂，参见文献 [207]。

[27] The locally supersymmetric sigma model for $E_{8(8)}/SO(16)$ has also be derived from dimensional reduction of $11D$ supergravity [210].

[27] $E_{8(8)}/SO(16)$ 的局域超对称 sigma 模型也可通过 $11D$ 超引力的维约化导出 [210]。

A proper subgroup G_0 of the isometry group G of the scalar manifold can be gauged by introducing an embedding tensor θ_{MN} which specifies G_0 and it is invariant under it. This is expressed by

标量流形等距群 G 的真子群 G_0 可以通过引入嵌入张量 θ_{MN} 来规范，该张量指定了 G_0 且在 G_0 下不变，可表示为

$$\theta_{PL}\left(f^{KL}{}_{M}\theta_{NK}+f^{KL}{}_{N}\theta_{MK}\right)=0, \tag{226}$$

where $\theta_{PL}f^{KL}{}_{M}$ are the structure constants of G_0 . Furthermore, the requirement of local supersymmetry of the action imposes the following constraint linear in the embedding tensor,

其中 $\theta_{PL}f^{KL}{}_{M}$ 是 G_0 的结构常数。此外，作用量的局域超对称要求对嵌入张量施加了如下线性约束:

$$(N-2)\left(T^{ij,k\ell}-T^{[ij,k\ell]}\right)+\left(\delta^{i[k}T^{\ell]m,mj}-i\leftrightarrow j\right)+\frac{2\delta^{i[k}\delta^{\ell]j}}{(N-1)}T^{mn,mn}=0,$$

(227)

where

其中

$$T^{ij,k\ell}=\mathcal{V}^{Mij}\theta_{MN}\mathcal{V}^{Nk\ell},\ \theta_{MN}=\theta_{NM}, \tag{228}$$

and $\mathcal{V}^{Mij}$ are moment maps associated with the Killing vector fields $X^{M\alpha}(\phi)$ that generate the isometry group satisfying the defining equation

且 $\mathcal{V}^{Mij}$ 是生成等距群的 Killing 向量场 $X^{M\alpha}(\phi)$ 对应的动量映射，满足定义方程

$$D_\alpha \mathcal{V}^{Mij}(\phi, X) = \frac{1}{2} f^{ij}_{\alpha\beta}(\phi) X^{M\beta}(\phi). \tag{229}$$

The Lie derivative of the complex structures, the Lie derivative of $SO(N)$ connection, and the complex structures $f^{ij}_{\alpha\beta}$ vanish up to a local $SO(N)$ rotation with parameter depending on (X, ϕ) . As for the condition (227), it also holds for $N = 1, 2$ in which case it degenerates to an identity [152].

复结构的李导数、$SO(N)$ 联络的李导数，以及复结构 $f^{ij}_{\alpha\beta}$，在依赖于 (X, ϕ) 的局域 $SO(N)$ 变换下都为零。至于条件 (227)，它对 $N = 1, 2$ 也成立，此时该条件退化为恒等式 [152]。

The bosonic part of Lagrangian is given by [152]

拉格朗日量的玻色子部分由下式给出 [152]

$$\mathcal{L} = -\frac{1}{2}\mathrm{i}\varepsilon^{\mu\upsilon\rho} e_\mu{}^a R_{\upsilon\rho a} - \frac{1}{2} e g_{\alpha\beta} D_\mu \phi^\alpha D_\mu \phi^\beta$$

$$+\frac{1}{4}\,\mathrm{i g}\,\varepsilon^{\mu\upsilon\rho} A^M_\mu \theta_{MN} \left(\partial_\upsilon A^N_\rho - \frac{1}{3} g \widehat{f}_{PQ}{}^N A^P_\upsilon A^Q_\rho\right)$$

$$+\frac{4eg^2}{N^2}\left(N A^{ij}_1 A^{ij}_1 - \frac{1}{2} g^{\alpha\beta} D_\alpha A^{ij}_1 D_\beta A^{ij}_1 - 2 g^{\alpha\beta} T^{ij}_\alpha T^{ij}_\beta\right), \tag{230}$$

where $R_{\mu\upsilon a} = \varepsilon_{abc} R_{\mu\nu}{}^{bc}$, and $D_\alpha A^{ij}_1 = \partial_\alpha A^{ij}_1 + 2 Q^{k[i}_\alpha A^{j]k}_1$ in which Q^{ij}_α is the $SO(N)$ target space connection, and

其中 $R_{\mu\upsilon a} = \varepsilon_{abc} R_{\mu\nu}{}^{bc}$，且 $D_\alpha A^{ij}_1 = \partial_\alpha A^{ij}_1 + 2 Q^{k[i}_\alpha A^{j]k}_1$，式中 Q^{ij}_α 是 $SO(N)$ 目标空间联络，且

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + g \theta_{MN} A^M_\mu X^{N\alpha}$$

$$(N-2) A^{ij}_1 = \mu (N-2) \delta^{ij} - 4 T^{i\ell, j\ell} + \frac{2}{N-1} T^{mn,mn} \delta^{ij},$$

$$T^{ij\alpha} = \mathcal{V}^{Mij} \theta_{MN} X^{N\alpha}$$

$$\widehat{f}_{MN}{}^P = \theta_{MQ} f^{PQ}{}_N \tag{231}$$

The supertransformations of the fermionic fields are

费米子场的超变换为

$$\delta \psi^i_\mu = D_\mu \varepsilon^i$$

$$\delta\chi^{\alpha i} = \frac{1}{2}\left(\delta^{ij} - f^{ij}\right)^{\alpha}{}_{\beta}\gamma^{\mu}D_{\mu}\phi^{\beta}\varepsilon^{j}, \tag{232}$$

where

其中

$$D_{\mu}\varepsilon^{i} = \left(\partial_{\mu} + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\varepsilon^{i} + \partial_{\mu}\phi^{\alpha}Q_{\alpha}^{ij}\varepsilon^{j} + g\theta_{MN}A_{\mu}^{M}\mathcal{V}^{Nij}\varepsilon^{j}. \tag{233}$$

Several properties of the scalar manifold as well as the solutions of the constraints on the embedding tensor θ_{PQ} for different values of N have been described in [152]. It is noteworthy that the scalar manifold is an arbitrary Riemannian manifold for $N=1$, Kähler for $N=2$, quaternionic for $N=3$, locally a product of two quaternionic manifolds for $N=4$ and symmetric homogeneous space G/H such that $d=\dim G-\dim H$ for $N>4$. The symmetric ones for $4\leq N\leq 16$ are listed in Table 6.

针对不同 N 取值，标量流形的若干性质以及嵌入张量 θ_{PQ} 的约束解已在文献 [152] 中得到阐述。值得注意的是: $N=1$ 对应的标量流形是任意黎曼流形，$N=2$ 对应凯勒流形，$N=3$ 对应四元数流形，$N=4$ 对应局部为两个四元数流形的乘积，$N>4$ 对应对称齐性空间 G/H 且满足 $d=\dim G-\dim H$ 。$4\leq N\leq 16$ 对应的对称流形已列于表 6。

Comments on Off-Shell Supergravities in 3D

三维离壳超引力评述

In $3D$, the off-shell $N=1$ supergravity has the fields $\left(e_{\mu}^{a}, S\right)$, where the auxiliary field S is a real scalar. The off-shell $N=(1,1)$ supergravity, [28] on the other hand, has the field content $\left\{e_{\mu}{}^{a}, \psi_{\mu}, V_{\mu}, S\right\}$ where the gravitino is Dirac, the auxiliary vector is non-gauge, and the auxiliary scalar S is complex. As to the off-shell $N=(2,0)$ supergravity, the field content is $\left\{e_{\mu}^{a}, \psi_{\mu}, C_{\mu}, V_{\mu}, D\right\}$ where the gravitino is Dirac, C_{μ} is gauge field, the auxiliary field V_{μ} is non-gauge and the auxiliary field D is real; see, for example, [214,215] for the Lagrangians and further details of these off-shell supergravities. Off-shell $N\leq 4$ matter coupled supergravities were constructed in superspace formalism in [216], and in AdS (p,q) superspace framework for $p+q\leq 4$ in [217].

在 $3D$ 中，离壳 $N=1$ 超引力包含场量 $\left(e_{\mu}^{a}, S\right)$，其中辅助场 S 是实标量场。另一方面，离壳 $N=(1,1)$ 超引力 ([28]) 的场内容为 $\left\{e_{\mu}{}^{a}, \psi_{\mu}, V_{\mu}, S\right\}$，其中引力微子是狄拉克费米子，辅助矢量场是非规范场，辅助标量 S 是复标量。至于离壳 $N=(2,0)$ 超引力，其场内容为 $\left\{e_{\mu}^{a}, \psi_{\mu}, C_{\mu}, V_{\mu}, D\right\}$，其中引力微子是狄拉克费米子，$C_{\mu}$ 是规范场，辅助场 V_{μ} 是非规范场，辅助场 D 是实场；这些离壳超引力的拉格朗日量和更多细节例如可参见文献 [214,215]。离壳 $N\leq 4$ 物质耦合超引力已在文献 [216] 中用超空间形式构造完成，AdS (p,q) 超空间框架下针对 $p+q\leq 4$ 的构造则在文献 [217] 中完成。

Lower Dimensions

低维

In one and two dimensions we get into the realm of infinite dimensional rigid symmetries and gaugings. It is technically quite a complicated matter to render these symmetries manifest. As this would take us far afield, we shall be content with only providing some basic facts and references.

在一维和二维中，我们进入了无穷维刚性对称性与规范对称领域。要让这些对称性显化在技术上相当复杂。由于这会偏离我们当前的主题，我们仅提供一些基础事实与参考文献就满足了。

$D = 2$

In $2D$, there exists (p,q) type of supersymmetry where p and q refer to the number of left- and right-handed supersymmetry generators. Couplings of scalar multiplets to $2D$ conformal supergravity were constructed long ago [218-223]. These are often referred to as locally supersymmetric sigma models. The geometry of the scalar manifold is Riemannian for $(1,0)$, hermitian with torsion for $(2,0)$, and hyperkähler or quaternionic Kähler for $(4,0)$.[29] In the case of $N = (1,0)$, taking the scalars to correspond to the coordinates of spacetime, and by inclusion of Wess-Zumino term and heterotic fermions, i.e., fermions which are singlets under supersymmetry, the model describes a heterotic string in curved background. In the case of $N = (8,0)$, the algebraic constraints on the scalar multiplet fields that follow from the supergravity multiplet equations of motion are solved to obtain $N = (8,0)$ supergravity coupled to $8n$ scalars which parametrize the coset $SO(n,8)/(SO(n) \times SO(8))$ [223] .

在 $2D$ 中，存在 (p,q) 型超对称，其中 p 和 q 分别对应左手征与右手征超对称生成元的个数。标量多重态与 $2D$ 共形超引力的耦合早已被构造出来 [218-223]，这类模型通常被称为局域超对称 sigma 模型。对于 $(1,0)$ ，标量流形的几何是黎曼几何；对于 $(2,0)$ ，是带挠率的埃尔米特几何；对于 $(4,0)$.[29] ，则是超凯勒或四元数凯勒几何。在 $N = (1,0)$ 的情形下，若将标量对应为时空坐标，再引入韦斯-祖米诺项和杂化费米子 (即超对称下的单态费米子)，该模型就描述了弯曲背景下的杂化弦。在 $N = (8,0)$ 的情形下，求解超引力多重态运动方程导出的标量多重场代数约束，可得到耦合了 $8n$ 标量的 $N = (8,0)$ 超引力，这些标量参数化陪集 $SO(n,8)/(SO(n) \times SO(8))$ [223] 。

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[28] The $N = (p,q)$ terminology in $3D$ refers to supergravities with the R -symmetry group $SO(p)\times SO(q)$ and admitting a vacuum solution with anti-de Sitter group, $OSp(2,p) \oplus OSp(2,q)$, symmetry.

[28] $3D$ 中的 $N = (p,q)$ 术语指的是满足以下条件的超引力: 其 R 对称群为 $SO(p)\times SO(q)$ ，且存在具有反德西特群 $OSp(2,p) \oplus OSp(2,q)$ 对称性的真空解。

[29] As observed in [222], the possibility of hyperkähler in the $(4,4)$, or $(4,0)$ cases is due to the fact that in $2D$ there is no gravitino kinetic term.

[29] 正如文献 [222] 中指出的，在 $(4,4)$ 或 $(4,0)$ 情形下存在超凯勒几何的原因是，$2D$ 中不存在引力微子动能项。

The simplest nonconformal coupling of supergravity to scalar multiplets has the action $S = \int d^2xe\phi(R - \Lambda)$ as its bosonic part, and it is known as Jackiw-Teitelboim (JT) gravity which has been a subject of numerous studies in recent years owing to its being a solvable model of gravity, with remarkable properties including its dual relation to random matrix models in the boundary of AdS_2 ; see [224] for a review. One generalization replaces Λ by a function $W(\phi)$. A general coupling of $N=(2,2), 2D$ supergravity to scalar multiplets was given in [225].

超引力与标量多重态最简单的非共形耦合的玻色子部分作用量为 $S=\int d^2xe\phi(R-\Lambda)$ ，即 Jackiw-Teitelboim(JT) 引力；它是可解的引力模型，具有包括与 AdS_2 边界随机矩阵模型的对偶关系在内的诸多优良性质，近年来得到了大量研究，综述参见 [224]。该模型的一个推广是将 Λ 替换为函数 $W(\phi)$ 。$N=(2,2), 2D$ 超引力与标量多重态的一般耦合已在文献 [225] 中给出。

The classical Lagrangian for the maximal $N=(8,8), 2D$ supergravity, often referred to as the $N=16$ supergravity, is easily obtained from a circle reduction of the maximal $3D$ supergravity in which the 128 physical scalars parametrize the coset $E_8/SO(16)$. The equations of motion of the $N=16, 2D$ supergravity have been shown to be integrable in the sense that they follow from a linear system of equations [226,227]. This theory has long been expected to have a hidden E_9 symmetry [228]. The group E_9 is based on the hyperbolic Kac-Moody algebra $\mathfrak{e}_9$, and it is the centrally extended loop group over E_8 , extended with the Virasoro generator L_0 . It turns out that there is an extension of this symmetry to $\hat{E}_8 \rtimes \mathrm{Vir}_-$ based on an algebra with the generators [229]

最大 $N=(8,8), 2D$ 超引力的经典拉格朗日量 (常被称为 $N=16$ 超引力) 很容易通过最大 $3D$ 超引力的圆约化得到，其中 128 个物理标量参数化了陪集 $E_8/SO(16)$ 。已经证明 $N=16, 2D$ 超引力的运动方程是可积的，在此意义上它们可由线性方程组导出 [226,227]。长期以来人们认为该理论存在一个隐藏的 E_9 对称性 [228]。群 E_9 基于双曲卡茨-穆迪代数 $\mathfrak{e}_9$ ，它是 E_8 上的中心扩张圈群，并由维拉索罗生成元 L_0 扩张。事实表明，该对称性可扩展为基于带生成元的代数的 $\hat{E}_8 \rtimes \mathrm{Vir}_-$ [229]

$$\underbrace{T_n^A\,(n\in\mathbb{Z}), K}_{\hat{E}_8}, \underbrace{L_n\,(n\leq 0)}_{\mathrm{Vir}^-},\ A=1,\dots,248. \tag{234}$$

For the algebra of these generators see, for example, [230, eqs. (2.5)-(2.7)]. The generators of $\hat{E}_8$ together with L_0 form the algebra that underlines E_9 . Denoting its maximal compact subgroup by $K(E_9)=K\left(\hat{E}_8\right)$, the bosonic fields of the $2D$ theory, all scalars, parametrize the coset [230, 231]

这些生成元的代数可参见例如文献 [230, 式 (2.5)-(2.7)]。$\hat{E}_8$ 的生成元与 L_0 共同构成了支撑 E_9 的代数。将其极大紧子群记为 $K(E_9)=K\left(\hat{E}_8\right)$ ，$2D$ 理论的玻色场 (全部为标量) 参数化了如下陪集 [230, 231]

$$\frac{\hat{E}_8\times\mathrm{Vir}^-}{K(E_9)}. \tag{235}$$

A building block for the description of the theory is the representative of this coset given by

该理论描述的一个构建块是如下给出的这个陪集的代表元

$$V=\underbrace{e^{-L_0}e^{-\varphi_1L_{-1}}e^{-\varphi_2L_{-2}}\cdots}_{\mathrm{Vir}^-}\underbrace{e^{-\sigma K}V_0e^{Y_{1A}T_{-1}^A}e^{Y_{2A}T_{-2}^A}\cdots}_{\hat{E}_8} \tag{236}$$

where ρ is the dilaton, $\varphi_n, n \geq 1$, are scalars dual to the dilaton, σ is the conformal scalar coming from the choice of conformal gauge $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}, V_0$ is the representative of the coset $E_8/SO(16)$, and Y_{nA} are dual to the physical scalars that parametrize the coset $E_8/SO(16)$.

其中 ρ 是 dilaton(dilation 场)，$\varphi_n, n \geq 1$ 是对偶于 dilaton 的标量，σ 是因共形规范选取产生的共形标量，$g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}, V_0$ 是陪集 $E_8/SO(16)$ 的代表元，Y_{nA} 对偶于参数化陪集 $E_8/SO(16)$ 的物理标量。

A systematic construction of the bosonic sector of gauged matter-coupled supergravity theories in $2D$ was initiated in [232] where the embedding tensor is determined to transform in the basic representation of E_9 . The complete bosonic dynamics of all gauged maximal supergravities that admit a geometric uplift to $10D$ and $11D$, including the construction of the full potential, was achieved recently in [230], by starting from the E_9 exceptional field theory based on the building blocks outlined above [231, 233], and performing a generalized Scherk-Schwarz dimensional reduction. Denoting the generators of the algebra underlying $\widehat{E}_8 \rtimes \mathrm{Vir}^-$ by T_α , an arbitrary gauging is described by an embedding tensor $\theta_M{}^\alpha$ which defines the generators,

$2D$ 中带规范耦合物质的超引力理论玻色子 sector 的系统构造最早始于文献 [232]，该文中确定了嵌入张量变换属于 E_9 的基础表示。所有可几何上隆到 $10D$ 和 $11D$ 的带规范最大超引力的完整玻色动力学 (包括全势的构造) 最近在文献 [230] 中完成，该工作从基于上述构建块的 E_9 例外场论出发 [231, 233]，执行了广义谢尔克-施瓦茨维数约化。将支撑 $\widehat{E}_8 \rtimes \mathrm{Vir}^-$ 的代数的生成元记为 T_α ，任意规范耦合都由嵌入张量 $\theta_M{}^\alpha$ 描述，该张量定义生成元:

$$X_M = \theta_M{}^\alpha T_\alpha, \tag{237}$$

such that $D_\mu = \partial_\mu - gA_\mu{}^M X_M$. In absence of a full supersymmetry analysis of $2D$ gauged supergravity, investigations of examples and analogy with higher-dimensional situations suggest that the embedding tensor $\theta_{M\alpha}$ is of the restricted form [230,232]

满足 $D_\mu = \partial_\mu - gA_\mu{}^M X_M$ 。在缺少对 $2D$ 带规范超引力完整超对称分析的情况下，对实例的研究和与高维情形的类比表明，嵌入张量 $\theta_{M\alpha}$ 具有受限形式 [230,232]

$$\theta_{M\alpha} = -\eta_{-1\alpha\beta}\theta_N T^{\beta N}{}_M, \tag{238}$$

where $\eta_{-1\alpha\beta}$ is defined below in Eq. (243). Given the coset representative V discussed above, the covariantized scalar current takes the form [230]

其中 $\eta_{-1\alpha\beta}$ 由下文式 (243) 定义。给定上述讨论的陪集代表元 V ，协变化标量流具有如下形式 [230]

$$P_\mu = \frac{1}{2}D_\mu V V^{-1} + \text{ h.c. } = \frac{1}{2}\left(\partial_\mu V V^{-1} - A_\mu{}^M \theta_M{}^\alpha V T_\alpha V^{-1}\right) + \text{ h.c. }, \tag{239}$$

The desired equations of motion are encoded in the duality equation

所需的运动方程被编码在对偶方程中

$$P_\mu^{(n)} = S_n\left(P_\mu\right) + \chi_{\mu n} K, \ n \in \mathbb{Z}, \tag{240}$$

where $S_n, n \in \mathbb{Z}$ are the shift operators acting as

其中 $S_n, n \in \mathbb{Z}$ 是作用为如下的移位算符

$$S_n\left(T_m^A\right) = T_{n+m}^A,\ S_n\left(L_m\right) = L_{n+m},\ S_n\left(K\right) = 0, \tag{241}$$

and $\chi_{\mu n}$ for $n > 0$ are independent auxiliary fields, introduced to achieve $K\left(E_9\right)$ covariance of the shifted currents. A pseudo-Lagrangian which gives the duality equation (240) under a suitable projection (hence the terminology of "pseudo-Lagrangian," since the full duality equation is to be imposed by hand), turns out to be [230]

而 $\chi_{\mu n}$ 和 $n > 0$ 是相互独立的辅助场，引入它们是为了使平移后的流满足 $K\left(E_9\right)$ 协变性。在合适投影下可给出对偶方程 (240) 的伪拉格朗日量 (因此得名"伪拉格朗日量"——完整对偶方程需要手动添加)，形式如下 [230]

$$\mathcal{L} = \mathcal{L}_{\text{top}} - \frac{1}{2\rho^3}\left\langle \theta \left| M^{-1} \right| \theta \right\rangle - \frac{1}{2\rho}\eta_{-2\alpha\beta}\left\langle \theta \left| T^\alpha M^{-1} T^\beta \right| \theta \right\rangle, \tag{242}$$

where $M = V^\dagger V$, and $\widehat{E}_8$ invariant symmetric bilinear forms η_k are defined by

其中 $M = V^\dagger V$ 和 $\widehat{E}_8$ 不变对称双线性型 η_k 定义为

$$\eta_{k\alpha\beta} T^\alpha \otimes T^\beta = \sum_{n\in\mathbb{Z}} \eta_{AB} T_n^A \otimes T_{k-n}^B - L_k \otimes K - K \otimes L_k, \tag{243}$$

and $\mathcal{L}_{\text{top}}$ is the topological Lagrangian, which has a rather complicate form, and is given in [230, eq. (4.51)]. The bra-ket notation here is simply denoting contraction of the matrix indices with vectors and their transpose. See [230] for further details.

而 $\mathcal{L}_{\text{top}}$ 是拓扑拉格朗日量，形式较为复杂，已给出在文献 [230, 式 (4.51)] 中。此处的狄拉克记号仅表示矩阵指标与向量及其转置的缩并，更多细节参见 [230]。

Finally let us not that infinite dimensional hidden symmetries naturally arise also in $2D$ supergravity in less than maximal supersymmetry. For example, see [234] for a treatment of the $O(8,24)$ loop group arising as a symmetry of the equations of motion of half-maximal $2D$ supergravity, obtained from the toroidal compactification of heterotic string theory.

最后我们指出，无限维隐藏对称性也会自然出现在非最大超对称的 $2D$ 超引力中。例如，关于半极大 $2D$ 超引力运动方程对称所对应的 $O(8,24)$ 圈群的研究，可参见 [234]，该理论来自杂化弦理论的环面紧致化。

D = 1

$N = 1, 1D$ supergravity multiplet consists of (e, ψ_t) , where e is the einbein and ψ_t is the gravitino. Its coupling to d scalar multiplets $\left(\phi^i, \chi^i\right), i = 1, \ldots, d$, where $\phi^i(\tau)$ and $\chi^i(\tau)$ are bosons and fermions, was constructed in [219] as an action for a spinning particle. Generalization of this model to N -extended locally supersymmetric model in which the global $SO(N)$ symmetry is gauged by introducing the gauge fields $f_{ab} = f_{[ab]}, a = 1, \ldots, N$ was constructed in [235], where it shown that upon quantization it provides a relativistic wave equation for spin $N/2$. For issues related to compatibility of curved space background with local supersymmetry, see [235, 236].

> $N = 1, 1D$ 超引力多重态由 (e, ψ_t) 构成，其中 e 是爱因拜因子，ψ_t 是引力微子。它与 d 个标量多重态 $(\phi^i, \chi^i), i = 1, \ldots, d$ 的耦合 (其中 $\phi^i(\tau)$ 和 $\chi^i(\tau)$ 分别为玻色子和费米子) 已在文献 [219] 中构造为旋转粒子的作用量。该模型推广到 N 扩展定域超对称模型 (其中整体 $SO(N)$ 对称性通过引入规范场 $f_{ab} = f_{[ab]}, a = 1, \ldots, N$ 被定域化) 已在文献 [235] 中完成，该文献证明量子化后该模型给出自旋 $N/2$ 的相对论波动方程。关于弯曲空间背景与定域超对称性的相容性问题，参见文献 [235, 236]。

$N = 16, 1D$ supergravity coupled to scalar multiplets whose scalars parametrize the coset $G/H = SO(9,9)/(SO(9) \times SO(9$ has been obtained by dimensional reduction of Type I supergravity on a torus T^9 [237]. The $1D$ theory has the bosons $(e, \varphi, \mathcal{V}_M^A)$ and the fermions $(\psi_{t\alpha}, \chi_\alpha, \chi_{\bar{i}\alpha})$, where $i, \bar{i} = 1, \ldots, 9$ label the $SO(N) \times SO(N)$ vectors, $\alpha = 1, \ldots, 16$ is the $SO(9)$ spinor index, and $\mathcal{V}_M^A = (\mathcal{V}_M^i, \mathcal{V}_M^{\bar{i}})$ is the representative of the coset G/H . The scalar currents are defined by

> $N = 16, 1D$ 超引力耦合到标量参数化陪集 $G/H = SO(9,9)/(SO(9) \times SO(9))$ 的标量多重态，是通过将 I 型超引力在环面 T^9 上维数约化得到的 [237]。该 $1D$ 理论包含玻色子 $(e, \varphi, \mathcal{V}_M^A)$ 和费米子 $(\psi_{t\alpha}, \chi_\alpha, \chi_{\bar{i}\alpha})$ ，其中 $i, \bar{i} = 1, \ldots, 9$ 标记 $SO(N) \times SO(N)$ 矢量，$\alpha = 1, \ldots, 16$ 是 $SO(9)$ 旋量指标，$\mathcal{V}_M^A = (\mathcal{V}_M^i, \mathcal{V}_M^{\bar{i}})$ 是陪集 G/H 的代表元。标量流定义为

$$\mathcal{V}^{-1}\partial_t \mathcal{V} = \frac{1}{2}Q^{ij}X^{ij} + \frac{1}{2}Q_{\bar{i}\bar{j}}X^{\bar{i}\bar{j}} + P_{i\bar{i}}Y^{i\bar{i}}, \tag{244}$$

where $(Q^{ij}, Q_{\bar{i}\bar{j}})$ is the $SO(9) \times SO(9)$ composite gauge fields, X_{ij} and $X_{\bar{i}\bar{j}}$ are the $SO(9) \times SO(9)$ generators, and $Y_{i\bar{i}}$ are the G/H coset generators. The bosonic part of the Lagrangian is given by [237]

> 其中 $(Q^{ij}, Q_{\bar{i}\bar{j}})$ 是 $SO(9) \times SO(9)$ 复合规范场，X_{ij} 和 $X_{\bar{i}\bar{j}}$ 是 $SO(9) \times SO(9)$ 生成元，$Y_{i\bar{i}}$ 是 G/H 陪集生成元。拉格朗日量的玻色子部分由 [237] 给出

$$\mathcal{L} = \frac{1}{4}e^{-1}\left(P_{i\bar{j}}P_{i\bar{j}} - \dot{\varphi}^2\right), \tag{245}$$

and the supertransformations by

> 超变换由下式给出

$$\delta\psi_{t\alpha} = D_t\varepsilon_\alpha,\ \delta\chi_\alpha = -\frac{1}{2}e^{-1}\varepsilon_\alpha\dot{\varphi},\ \chi_{\bar{j}\alpha} = -\frac{1}{2}e^{-1}(\gamma_i\varepsilon)_\alpha P_{i\bar{j}}. \tag{246}$$

The dimensional reduction of $11D$ supergravity to $1D$ was conjectured long ago to have an infinite dimensional E_{10} symmetry [238], which is a hyperbolic Kac-Moody algebra. This reduction was carried out in [210] for the bosonic sector, first by a toroidal reduction to $3D$, then a circle reduction to $2D$ followed by a null reduction to $1D$. A nonlinear σ -model in $1D$ based on a coset $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of E_{10} , was proposed in [239]. The model was extended to include the fermions in

[240], where relation to $11D$ supergravity was studied as well. For a detailed treatment of the relation between the $E_{10}/K(E_{10})$ coset model, which requires an infinite set of fields most conveniently described in an $SO(9) \times SO(9)$ basis, and the $1D$ supergravity action discussed above, see [237]. See also the review [241], and reference therein.

人们很早就猜想，将 $11D$ 超引力维度约化到 $1D$ 后会存在无穷维 E_{10} 对称性 [238]，该对称性是一个双曲卡茨-穆迪代数。文献 [210] 对玻色子部分完成了这一约化过程: 先进行环面约化得到 $3D$，再进行圆周约化得到 $2D$，最后做类零约化得到 $1D$。文献 [239] 提出了 $1D$ 中基于陪集 $E_{10}/K(E_{10})$ 的非线性 σ 模型，其中 $K(E_{10})$ 是 E_{10} 的极大紧子群。文献 [240] 将该模型推广到包含费米子的情况，同时也研究了它与 $11D$ 超引力的关系。关于 $E_{10}/K(E_{10})$ 陪集模型 (它需要一组无穷多的场，在 $SO(9) \times SO(9)$ 基下描述最为方便) 与上述 $1D$ 超引力作用量之间关系的详细处理，参见文献 [237]，也可参见综述文献 [241] 及其中的参考文献。

Supergravities in Extended Geometry Framework

扩展几何框架中的超引力

It is well known that the U -duality groups $E_{n(n)}$ arise in maximal supergravities in D dimensions, where $n = 11 - D$, as an enhancement of the manifest $GL(n, \mathbb{R})$ symmetry in the reduction of $11D$ supergravity on n -torus. In the case of $N = 1, 10D$ supergravity coupled to n_V abelian vector multiplets, the enhanced symmetry is $O(n, n + n_V)$ which generates the T -duality group. In the case of non-abelian vectors, $O(n, n + n_V)$ symmetry is broken but its covariance is maintained, as in gauged supergravities. An extended geometrical framework has been developed in which the above mentioned symmetries/covariance are manifest prior to any reduction. Such theories in the case of E_n symmetries are called Exceptional Field Theories (ExFT), and in the case of $O(n, n + n_V)$ symmetry they are known as Double Field Theories (DFT). See the reviews [242-244] and the references therein. To achieve these symmetries, extra internal space coordinates are introduced in ExFTs, and the spacetime doubling occurs as well in DFTs, together with the so-called section constraints to ensure the closure of the generalized diffeomorphisms. Depending on how these constraints are solved, the ExFTs can be shown to be fully equivalent to $11D$ supergravity or type IIB supergravity, and the $N = 1$ supersymmetric DFT equivalent to $N = 1, 10D$ heterotic supergravity, as will be discussed further below.

众所周知，U 对偶群 $E_{n(n)}$ 出现在 D 维最大超引力中，其中 $n = 11 - D$ 是 $11D$ 超引力约化到 n 环面后，显然的 $GL(n, \mathbb{R})$ 对称性的增强。对于 $N = 1, 10D$ 超引力耦合 n_V 个阿贝尔矢量多重态的情况，增强对称性为 $O(n, n + n_V)$，它生成了 T 对偶群。对于非阿贝尔矢量的情况，$O(n, n + n_V)$ 对称性会发生破缺，但仍保留协变性，这和 gauged 超引力中的情况一致。目前已经发展出了扩展几何框架，使得上述对称性/协变性在任何约化之前就可以显然呈现。这类理论在具有 E_n 对称性时被称为例外场论 (ExFT)，而在具有 $O(n, n + n_V)$ 对称性时被称为双场论 (DFT)。相关综述可参见文献 [242-244] 及其中引文。为实现这些对称性，例外场论中引入了额外的内空间坐标，双场论中也对时空做了加倍，同时引入了所谓的截面约束来保证广义微分同胚的封闭性。根据这些约束的不同求解方式，可以证明例外场论完全等价于 $11D$ 超引力或 IIB 型超引力，$N = 1$ 超对称双场论等价于 $N = 1, 10D$ 杂化超引力，下文会对此做进一步讨论。

ExFTs associated with maximal supergravities have been constructed for $8 \geq D \geq 2$ [231,245-253] and a proposal exists for a so-called master exceptional field theory based on E_{11} [23]. ExFTs associated with

$6D, N=(1,0)$ magical supergravities have been constructed in [254], and the $N=1$ supersymmetric DFT in [255]. We shall briefly summarize ExFTs based on $E_{n(n)}$ for $n=6,7,8$, and the $N=1$ supersymmetric DFT, since only for these cases the fermionic sectors have also been constructed so far [246, 248, 249, 255]. Among many uses of exceptional field theory formalism, it is worth mentioning that it is very efficient in solving the long outstanding problems of consistent truncations around various background geometries [256], and it also provides a remarkably powerful tool for computing the Kaluza-Klein mass spectra around these backgrounds [257].

与最大超引力相关的例外场论已经针对 $8 \geq D \geq 2$ 构造出来 [231,245-253]，并且目前已经存在一个基于 E_{11} 的所谓主例外场论方案 [23]。与 $6D, N=(1,0)$ 魔术超引力相关的例外场论已经在 [254] 中构造完成，$N=1$ 超对称双场论也已经在 [255] 中构造完成。我们将简要总结基于 $E_{n(n)}$ 、适用于 $n=6,7,8$ 的例外场论，以及 $N=1$ 超对称双场论，因为到目前为止只有这些情况完成了费米子 sector 的构造 [246, 248, 249, 255]。在例外场论形式体系的众多应用中，值得一提的是它非常擅长解决各类背景几何下长期悬而未决的一致截断问题 [256]，同时它也为计算这些背景下卡鲁扎-克莱因质量谱提供了一个格外强大的工具 [257]。

Extended Geometry

扩展几何

In the case of ExFTs the extended geometry is formulated in $(D+\dim R_1)$ dimensional generalized space coordinatized by $\left(x^\mu, Y^M\right)$, where R_1 is a suitable representation of $E_{n(n)}$. The x -space is referred to as the external space, and Y - space as the internal space. In this geometry the tensors are tensors under $E_{n(n)} \times \mathbb{R}$, with $\mathbb{R}$ representing the trombone symmetry. The generalized diffeomorphism with parameter Λ^M is supplemented by and extra local $\sum$ -symmetry with parameter $\sum_M$ in the case of $n=8$, and it acts on a generalized vector as [258-260]

对于例外场论 (ExFTs)，扩展几何在以 $\left(x^\mu, Y^M\right)$ 为坐标的 $(D+\dim R_1)$ 维广义空间中表述，其中 R_1 是 $E_{n(n)}$ 的一个合适表示。x 空间被称为外空间，Y 空间被称为内空间。在该几何中，张量都是 $E_{n(n)} \times \mathbb{R}$ 下的张量，其中 $\mathbb{R}$ 对应长号对称性。对于 $n=8$ ，参数为 Λ^M 的广义微分同胚会额外补充一个参数为 $\sum_M$ 的定域 $\sum$ 对称性，它作用在广义向量上的形式为 [258-260]

$$\delta V^M = \mathcal{L}_{(\Lambda,\sum)} V^M = \Lambda^N \partial_N V^M - \alpha_n P^M_{(adj)N,}{}^P{}_Q \partial_P \Lambda^Q V^N + \beta_n \partial_K \Lambda^K V^M$$

$$-\delta_{n,8} \sum\nolimits_P f^{PM}{}_Q V^Q, \tag{247}$$

where $P^M_{(adj)N},{}^P{}_Q$ is projector onto the adjoint representation of $E_{n(n)}$, and $\alpha_n = 6{,}12{,}60$for $n=6,7,8$, respectively, and $\beta_n = \frac{1}{9-n}$ which is the trombone weight of a vector field in $D=11-n$ dimensions. The last term in (247), where f_{MNP} are the structure constants of E_8 , is needed because when $n=8$ the Λ - transformations close only up to terms that can be interpreted as $\sum$ -transformations. Gauge connections $\left(A^M_\mu, B_{\mu M}\right)$ will be associated with the $(\Lambda, \sum)$ transformations. In the conventions of [245, 247, 249] , the projectors P_{adj} can be written as [30]

其中 $P^{M}_{(adj)N,}{}^{P}{}_{Q}$ 是到 $E_{n(n)}$ 伴随表示的投影算子，$\alpha_n =$ 分别对应 $n = 6,7,8$ 的 6、12、60，$\beta_n = \frac{1}{9-n}$ 是 $D = 11 - n$ 维中向量场的长号权。(247) 式的最后一项中 f_{MNP} 是 E_8 的结构常数，该项是必需的，因为当 $n = 8$ 时，Λ 变换只能差一个可解释为 $\sum$ 变换的项才能闭合。规范联络 $\left(A^{M}_{\mu}, B_{\mu M}\right)$ 会与 $(\Lambda, \sum)$ 变换关联。在 [245, 247, 249] 的约定下，投影算子 P_{adj} 可写为 [30]

$$P^{M}_{(adj)N,}{}^{P}{}_{Q} = \begin{cases} (t_\alpha)^{M}{}_{N}(t^\alpha)^{P}{}_{Q} & \text{for } n = 6,7 \\ \frac{1}{60} f^{M}{}_{NL} f^{LP}{}_{Q} & \text{for } n = 8 \end{cases}, \tag{248}$$

where t_α are the generators of $E_{6(6)}$ and $E_{7(7)}$ in their fundamental representations, and $f_{IJ}{}^{K}$ are the structure constants of $E_{8(8)}$. The definitions of covariant field strengths turn out to require the following extra fields:

其中 t_α 是 $E_{6(6)}$ 和 $E_{7(7)}$ 基础表示的生成元，$f_{IJ}{}^{K}$ 是 $E_{8(8)}$ 的结构常数。协变场强的定义证明需要引入以下额外场:

$$E_{6(6)}: \; B_{\mu\upsilon M},\; M = 1,\ldots,27,$$

$$E_{7(7)}: \; \left(B_{\mu\upsilon\alpha}, B_{\mu\upsilon M}\right),\; \alpha = 1,\ldots,133,\; M = 1,\ldots,56,$$

$$E_{8(8)}: \; \left(C_{\mu\upsilon}{}^{MN}, C_{\mu\upsilon M}{}^{N}, C_{\mu\upsilon}\right),\; M,N = 1,\ldots,248, \tag{249}$$

where the two-form $C_{\mu\upsilon}{}^{MN}$ is valued in the 3875 dimensional representation of E_8. It has been established that the transformations (247) close on all the fields,

其中二次型 $C_{\mu\upsilon}{}^{MN}$ 属于 E_8 的 3875 维表示。现已证明，变换 (247) 在所有场上闭合,

$$\left[\mathcal{L}_{(\Lambda_1,\sum_1)}, \delta_{(\Lambda_2,\sum_2)}\right] = \delta_{(\Lambda_{12},\sum_{12})},\; \left(\Lambda_{12}, \sum\nolimits_{12}\right) \equiv \left[\left(\Lambda_1, \sum\nolimits_1\right), \left(\Lambda_2, \sum\nolimits_2\right)\right]_E,$$

(250)provided that the following section constraints are imposed [245, 247, 249]

前提是需要满足以下截面约束 [245, 247, 249]

$$E_{6(6)}: \; d^{PMN}\partial_M \otimes \partial_N = 0,$$

$$E_{7(7)}: \; (t_\alpha)^{MN} C_M \otimes C'_N = 0,\; \Omega^{MN} C_M \otimes C'_N = 0,$$

$$C_M, C'_M \in \left\{\partial_M, B_{\mu\upsilon M}\right\}$$

$$E_{8(8)}: \; \eta^{MN} C_M \otimes C'_N = 0, f^{PMN} C_M \otimes C'_N = 0, \left(P_{(3875)}\right)_{PQ}{}^{MN} C_M \otimes C'_N = 0,$$

$$C_M, C'_M \in \left\{\partial_M, B_{\mu M}, \sum\nolimits_M\right\} \tag{251}$$

[30] The Cartan-Killing metrics $\kappa^{\alpha\beta} = (t^\alpha)^M{}_N (t^\beta)_N{}^M$, and $\eta^{MN} = \frac{1}{60} f^{MK}{}_L f^{NL}{}_K$ can be used to raise and lower indices.

[30] 嘉当-基林度量 $\kappa^{\alpha\beta} = (t^\alpha)^M{}_N (t^\beta)_N{}^M$ 和 $\eta^{MN} = \frac{1}{60} f^{MK}{}_L f^{NL}{}_K$ 可用于升降指标。

where d^{MNP} is the totally symmetric invariant tensor of E_6 , and Ω^{MN} is the antisymmetric invariant tensor of E_7 and η^{MN} is the symmetric invariant tensor of E_8 , and

其中 d^{MNP} 是 E_6 的全对称不变张量，Ω^{MN} 是 E_7 的反对称不变张量，η^{MN} 是 E_8 的对称不变张量，且

$$\left(P_{(3875)}\right)^{MK}{}_{NL} = \frac{1}{7}\delta^M_{(N}\delta^K_{L)} - \frac{1}{56}\eta^{MK}\eta_{NL} - \frac{1}{14} f^P{}_N{}^{(M} f_{PL}{}^{K)}, \tag{252}$$

which projects to the 3875 dimensional representation of E_8 . The E -bracket is defined for E_6, E_7, E_8 exceptional field theories in [245, 247, 249] , respectively. Schematically, the only nonvanishing commutators are of the form $[\Lambda, \Lambda] \sim \Lambda$ and $[\Lambda, \sum] \sim \sum$. The solutions of these constraints will be commented on in the following subsections. The gauge covariant derivatives on any tensors are naturally given by

它投影到 E_8 的 3875 维表示。E 括号分别对 [245, 247, 249] 中的 E_6, E_7, E_8 例外场论定义。概略来说，唯一非零对易子形如 $[\Lambda, \Lambda] \sim \Lambda$ 和 $[\Lambda, \sum] \sim \sum$ 。这些约束的解将在后续小节讨论。任意张量上的规范协变导数可自然表示为

$$D_\mu = \partial_\mu - \mathcal{L}_{(A_\mu, B_\mu)}. \tag{253}$$

The construction of the covariant field strengths for $\left(A^M_\mu, B_{\mu M}\right)$, and the fields listed in (249), requires the building of an exceptional field theory analog of the ordinary tensor hierarchy outlined in Appendix B. This needs to be done case by case in each spacetime dimension. In the following subsections, we shall note the field strengths for the $(\Lambda, \sum)$ gauge fields. Their Bianchi identities provide the definitions of higher form field strengths.

为了构造 $\left(A^M_\mu, B_{\mu M}\right)$ 以及式 (249) 所列场的协变场强，需要构建附录 B 中普通张量分层的例外场论类比。这需要在每个时空维数下逐个情况处理。在后续小节中，我们会给出 $(\Lambda, \sum)$ 规范场的场强。它们的比安基恒等式给出了高阶形式场强的定义。

In a theory formulated in the extended geometrical framework, in addition to the internal generalized diffeomorphisms with parameter $\Lambda(x, Y)$, there are also external diffeomorphisms with parameter $\xi^\mu(x, Y)$. In the universal sector, they take the form [245, 247, 249]

在扩展几何框架下表述的理论中，除了参数为 $\Lambda(x, Y)$ 的内广义微分同胚，还存在参数为 $\xi^\mu(x, Y)$ 的外微分同胚。在通用扇区中，它们形如 [245, 247, 249]

$$\delta e_\mu{}^a = \xi^\mu D_\nu e_\mu{}^a + D_\mu \xi^\nu e_\nu{}^a,$$

$$\delta \mathcal{M}_{MN} = \xi^\mu D_\mu \mathcal{M}_{MN}$$

$$\delta A_\mu^M = \begin{cases} \xi^\upsilon \mathcal{F}_{\upsilon\mu}{}^M + \mathcal{M}^{MN} g_{\mu\upsilon} \partial_N \xi^\upsilon & \text{for } n = 6,7 \\ \varepsilon_{\mu\upsilon\rho} \xi^\upsilon J^{\rho M} + \mathcal{M}^{MN} g_{\mu\upsilon} \partial_N \xi^\upsilon & \text{for } n = 8 \end{cases} \tag{254}$$

where $\mathcal{M}_{MN}$ is the generalized metric in the internal space, the duality equation $\xi^\upsilon \mathcal{F}_{\upsilon\mu}{}^M = \varepsilon_{\mu\upsilon\rho} \xi^\upsilon J^{\rho M}$ has been used in the case of $n = 8$, and

其中 $\mathcal{M}_{MN}$ 是内空间中的广义度规，对于 $n = 8$ 的情况已经使用了对偶方程 $\xi^\upsilon \mathcal{F}_{\upsilon\mu}{}^M = \varepsilon_{\mu\upsilon\rho} \xi^\upsilon J^{\rho M}$ ，且

$$D_\mu e_\upsilon{}^a = \partial_\mu e_\upsilon{}^a - A_\mu{}^M \partial_M e_\upsilon{}^a - \beta_n \left(\partial_M A_\mu{}^M \right) e_\upsilon{}^a. \tag{255}$$

For the transformations of the remaining fields under the external diffeomorphisms, see [249]. It is a remarkable fact that the external diffeomorphisms fix all the relative coefficients of the bosonic terms in the actions constructed so far, thereby producing the bosonic sectors of maximal supergravities without using local supersymmetry.

其余场在外微分同胚下的变换参见文献 [249]。一个值得注意的结论是，外微分同胚固定了迄今为止构造的作用量中所有玻色项的相对系数，因此无需借助局域超对称就可以得到最大超引力的玻色子 sector。

In closing this subsection, it is also useful to give the following universal formulae that provide a universal sector of the actions constructed. Firstly, the generalized Ricci scalar is defined as

在本小节最后，给出以下通用公式也很有用，这些公式给出了所构造作用量的通用扇区。首先，广义里奇标量定义为

$$\widehat{R} = e e_a{}^\mu e_b{}^\upsilon \left(R_{\mu\upsilon}{}^{ab}(\omega) + \mathcal{F}_{\mu\upsilon}{}^M e^{a\rho} \partial_M e_\rho{}^b \right). \tag{256}$$

The Riemann tensor $R_{\mu\nu ab}(\omega)$ has the standard expression in terms of the vielbein with all derivatives covariantized as in (255), and the second term is needed to ensure covariance not only under the generalized diffeomorphisms but also under the local Lorentz transformations [261]. Second, the potential which arises in the Lagrangians for the exceptional field theories is given by [245, 247, 249]

黎曼张量 $R_{\mu\nu ab}(\omega)$ 按标架有标准表达式，所有导数都如式 (255) 一样协变化，添加第二项是为了保证该张量不仅在广义微分同胚下协变，也在局域洛伦兹变换下协变 [261]。其次，例外场论拉格朗日中的势由下式给出 [245, 247, 249]

$$V = -\frac{1}{4\alpha_n} \mathcal{M}^{MN} \left(\partial_M \mathcal{M}^{KL} \right) \left(\partial_N \mathcal{M}_{KL} \right) + \frac{1}{2} \mathcal{M}^{MN} \left(\partial_M \mathcal{M}^{KL} \right) \left(\partial_L \mathcal{M}_{NK} \right)$$

$$-\frac{1}{2} \left(g^{-1} \partial_M g \right) \partial_N \mathcal{M}^{MN} - \frac{1}{4} \mathcal{M}^{MN} \left(g^{-1} \partial_M g \right) \left(g^{-1} \partial_N g \right) - \frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\upsilon} \partial_N g_{\mu\upsilon}$$

$$+\frac{1}{7200}\delta_{n,8}f^{NQ}{}_{P}f^{MS}{}_{R}\left(\mathcal{M}^{PK}\partial_{M}\mathcal{M}_{QK}\right)\left(\mathcal{M}^{RL}\partial_{N}\mathcal{M}_{SL}\right), \tag{257}$$

where the symmetric matrix $\mathcal{M}_{MN}$ is the "generalized metric" on the coset $E_{n(n)}/K_n$.

其中对称矩阵 $\mathcal{M}_{MN}$ 是陪集 $E_{n(n)}/K_n$ 上的“广义度规”。

Explicit solutions to the section constraints have been found [245] which yield $11D$ or type IIB supergravity compactified on n -torus but with dependence on the toroidal coordinates kept. Key to these solutions is the decomposition of the R_1 representation under $SL(n)\times GL(1)$ for the $11D$ embedding, and under $SL(n-1)\times SL(2)$ for the type IIB embedding, which are given by [245, 247, 247]

截面约束已经存在显式解 [245]，这些解给出约化到 n 环面的 $11D$ 或 IIB 型超引力，且保留了对环面坐标的依赖。对于 $11D$ 嵌入，解的核心是 R_1 表示在 $SL(n)\times GL(1)$ 下的分解，对于 IIB 嵌入则是 R_1 表示在 $SL(n-1)\times SL(2)$ 下的分解，分解结果为 [245, 247, 247]

$$11D：E_{n(n)}\to SL(n)\times GL(1)：R_1\to n_q+\cdots, \tag{258}$$

$$\text{Type IIB: } E_{n(n)}\to SL(n-1)\times GL(1)\times SL(2)：$$

$$R_1\to(n-1,1)_q+\cdots,$$

where q is the highest $GL(1)$ charge. The section constraints are then solved by (a) restricting the Y -dependence of all fields, including gauge parameters, to the n coordinates in the $11D$ embedding, or $n-1$ coordinates in the type IIB embedding, and (b) keeping only $B_{\mu\nu m}$ and $B_{\mu m}$ components of $B_{\mu\nu M}$ and $B_{\mu M}$ nonzero, in the cases of $E_{7(7)}$ and $E_{8(8)}$, respectively. Thus, a generic field $\Phi\left(x^\mu,Y^M\right)\to\ \Phi\left(x^\mu,y^m\right)$, where y^m corresponds to the n_q or $(n-1,1)_q$ representation. In the case of $E_{6(6)}$, it has been shown that $11D$ supergravity on 5-torus, with the toroidal coordinates kept, is equivalent to the $E_{6(6)}$ exceptional field theory with the section constraint solved as described above [245].

其中 q 是最高 $GL(1)$ 荷。随后截面约束可通过以下方式求解:(a) 将包括规范参数在内的所有场对 Y 的依赖限制为 $11D$ 嵌入中的 n 坐标，或 IIB 型嵌入中的 $n-1$ 坐标，(b) 在 $E_{7(7)}$ 和 $E_{8(8)}$ 情形下，分别仅保留 $B_{\mu\nu M}$ 和 $B_{\mu M}$ 的 $B_{\mu\nu m}$ 分量与 $B_{\mu m}$ 分量非零。因此，任意场 $\Phi\left(x^\mu,Y^M\right)\to\ \Phi\left(x^\mu,y^m\right)$ 中 y^m 对应 n_q 或 $(n-1,1)_q$ 表示。对于 $E_{6(6)}$ ，已有研究表明: 保留环面坐标的 5 环面上 $11D$ 超引力，等价于按上述方法求解截面约束后的 $E_{6(6)}$ 例外场论 [245]。

In the case of DFTs, focusing on the manifestly duality invariant formulation of $N=1,10D$ supergravity, the extended geometry is formulated in terms of the coordinates $X^M=\left(\tilde{x}_\mu,x^\mu,y^m\right)$ which are in the fundamental representation of the duality group $G=O(10,10+n_V)$, with $\mu,\tilde{\mu}=0,1,\ldots,9$ and $m=\ 1,\ldots,n_V$. In addition to this global symmetry, the theory has local double Lorentz $H=O(9,1)\times O(1,9+n_V)$ symmetry. The generalized diffeomorphism with parameter ξ^M acts on a generalized vector as

对于双场论 (DFT)，聚焦于 $N=1,10D$ 超引力的明显对偶不变表述，扩展几何基于坐标 $X^M = (\tilde{x}_\mu, x^\mu, y^m)$ 构造，这些坐标属于对偶群 $G = O(10,10+n_V)$ 的基础表示，对应 $\mu, \widetilde{\mu} = 0,1,\dots,9$ 和 $m = 1,\dots,n_V$。除该整体对称性外，该理论还具有局部双洛伦兹 $H = O(9,1)\times O(1,9+n_V)$ 对称性。带参数 ξ^M 的广义微分同胚作用在广义矢量上形式为

$$\delta V^M = \xi^N \partial_N V^M - \left(\partial^M \xi_N - \partial_N \xi^M\right) V^N + f^M{}_{NP} \xi^N V^P, \tag{259}$$

where $M = 0,1,\dots,19+n_V$. The constants $f_{MNP} = f_{MN}{}^L \eta_{LP}$, often referred to as fluxes or gaugings, break the G-invariance but the formalism is G-covariant, just as in the gauged supergravity theories. In DFT there also exists a dilaton field which transforms as

其中 $M = 0,1,\dots,19+n_V$。常数 $f_{MNP} = f_{MN}{}^L \eta_{LP}$ 常被称为流或规范耦合，它会破缺 G 不变性，但该形式体系仍然是 G 协变的，这与有规范超引力理论的情况一致。在双场论中还存在一个伸缩子场，其变换规律为

$$\delta d = \xi^M \partial_M d - \frac{1}{2}\partial_M \xi^M. \tag{260}$$

The algebra of these transformations closes provide that the following constraints are imposed

这些变换的代数在满足以下约束时闭合

$$\partial_M \partial^M \cdots = 0,\ \partial_M \cdots \partial^M \cdots = 0,\ f_{MN}{}^P \partial_P = 0,$$

$$f_{MNP} = f_{[MNP]},\ f_{[MN}{}^L f_{P]L}{}^Q = 0, \tag{261}$$

where the ellipses refer to any fields or gauge parameters.

其中省略号指代任意场或规范参数。

$E_{6(6)}$ Exceptional Supergravity in $(5+27_c)$ Dimensions

$E_{6(6)}$ 例外超引力，维度为 $(5+27_c)$

The bosonic sector was constructed in [245], and the supersymmetric version in [246]. We shall summarize the key results below based on these papers. The field content is given by

玻色子部分由文献 [245] 构造，超对称版本由文献 [246] 构造。我们将基于这两篇论文在下文中总结核心结果。场内容由下式给出:

$$\left(g_{\mu\nu}, \mathcal{V}_M{}^{ij}, A_\mu^M, B_{\mu\nu M}; \psi_\mu^i, \chi^{ijk}\right), \tag{262}$$

where $M = 1,\dots,27$ labels the fundamental representation of $E_{6(6)}$, and $\mathcal{V}_M{}^{ij}$, where $i = 1,\dots,8$ labels the fundamental representation of $USp(8)$, parametrizes the coset $E_{6(6)}/USp(8)$. The spinors are symplectic

Majorana. The two-form potentials have been introduced as duals of the gauge fields. The formulae summarized in the previous section apply. In the conventions of [245], the covariant two-form field strength is given by

其中 $M=1,\ldots,27$ 标记了 $E_{6(6)}$ 的基础表示，而标记了 $USp(8)$ 基础表示的 $\mathcal{V}_M{}^{ij}$ 和 $i=1,\ldots,8$ 对陪集 $E_{6(6)}/USp(8)$ 进行参数化。旋量是辛马约拉纳旋量。二形式势是作为规范场的对偶引入的。前一节总结的公式均适用。在文献 [245] 的约定下，协变二形式场强由下式给出

$$\mathcal{F}_{\mu\upsilon}{}^{M}=2\partial_{[\mu}A_{\upsilon]}^{M}-\left[A_{\mu},A_{\upsilon}\right]_{E}^{M}+10d^{MNK}\partial_{N}B_{\mu\upsilon K}. \tag{263}$$

The field strength $\mathcal{H}_{\mu\upsilon\rho M}$ can be read off from the Bianchi identity,

场强 $\mathcal{H}_{\mu\upsilon\rho M}$ 可从比安基恒等式直接读出,

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M}=\frac{10}{3}d^{MNK}\partial_{K}\mathcal{H}_{\mu\nu\rho N},\ D_{\mu}=\partial_{\mu}-\mathcal{L}_{A_{\mu}}. \tag{264}$$

We also need the scalar current and composite connection defined as

我们还需要如下定义的标量流和复合联络:

$$\mathcal{P}_{\mu}^{ijk\ell}=D_{\mu}\mathcal{V}_{M}^{[ij]}\mathcal{V}^{k\ell]M},\ \mathcal{Q}_{\mu i}{}^{j}=\frac{1}{3}\mathcal{V}_{ik}{}^{M}D_{\mu}\mathcal{V}_{M}{}^{jk}. \tag{265}$$

With the above ingredients at hand, the bosonic part of the Lagrangian takes the form [246]

有了上述要素之后，拉格朗日量的玻色部分可写为 [246]:

$$e^{-1}\mathcal{L}=\widehat{R}-\frac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}^{M}\mathcal{F}^{\mu\nu N}-\frac{1}{6}\mathcal{P}_{\mu}{}^{ijk\ell}\mathcal{P}^{\mu}{}_{ijk\ell}-V(\mathcal{M},g)+\frac{\sqrt{10}}{8}e^{-1}\mathcal{L}_{\text{top}}, \tag{266}$$

with $(\widehat{R},V)$ from (256) and (257), respectively, and $\mathcal{M}_{MN}=\mathcal{V}_{M}{}^{ij}\mathcal{V}_{Nij}$. The topological Lagrangian is given in [246], where its general variation is also given as

其中 $(\widehat{R},V)$ 分别来自式 (256) 和 (257)，且 $\mathcal{M}_{MN}=\mathcal{V}_{M}{}^{ij}\mathcal{V}_{Nij}$ 。拓扑拉格朗日量见文献 [246]，该文献也给出了它的全变分形式:

$$\delta\mathcal{L}_{\text{top}}=\varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{3}{4}d_{MNK}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma}{}^{N}\delta A_{\tau}{}^{K}+5d^{MNK}\partial_{N}\mathcal{H}_{\mu\upsilon\rho M}\Delta B_{\sigma\tau K}\right), \tag{267}$$

with the covariant variation $\Delta B_{\mu\upsilon M}\equiv\delta B_{\mu\upsilon M}+d_{NKL}A_{[m}{}^{K}\delta A_{\upsilon]}{}^{L}$. Using this formula, it is easy to derive the following (projected) duality equation as the field equation of $B_{\mu\upsilon M}$:

其中协变变为 $\Delta B_{\mu\upsilon M}\equiv\delta B_{\mu\upsilon M}+d_{NKL}A_{[m}{}^{K}\delta A_{\upsilon]}{}^{L}$ 。利用该公式，我们可以很容易地推导出如下 (投影后的) 对偶方程，它就是 $B_{\mu\upsilon M}$ 的场方程:

$$d^{MNK}\partial_K\left(\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}+\frac{\sqrt{10}}{6}\varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{H}_{\rho\sigma\tau M}\right)=0. \tag{268}$$

This ensures the correct $128_B + 128_F$ physical degrees of freedom. Finally, the supertransformations of the fermionic fields are [246]

这保证了 $128_B + 128_F$ 的物理自由度数目正确。最后，费米子场的超变换为 [246]:

$$\delta\psi_\mu = \mathcal{D}_\mu\varepsilon^i - i\sqrt{2}\mathcal{V}^{ijM}\left(\Omega_{jk}\nabla^-_M\left(\gamma_\mu\varepsilon^k\right)-\frac{1}{3}\gamma_\mu\nabla^-_M\varepsilon_j\right),$$

$$\delta\chi^{ijk} = \frac{i}{2}\mathcal{P}_\mu{}^{ijk\ell}\Omega_{\ell m}\gamma^\mu\varepsilon^m + \frac{3}{\sqrt{2}}\mathcal{V}^{\left[ijM\nabla^-_M\varepsilon^k\right]}, \tag{269}$$

where $[\![\,]\!]$ means totally antisymmetric and symplectic traceless, $\mathcal{D}_\mu = \mathcal{D}_\mu\left(A_v,\omega_v,\mathcal{Q}_v\right)$ and $\nabla^-_M = \nabla_M\left(\omega^-_N,\mathcal{Q}_n,\Gamma_N\right)$ with Γ_N the Christoffel connection and

其中 $[\![\,]\!]$ 表示全反对称且辛无迹，$\mathcal{D}_\mu = \mathcal{D}_\mu\left(A_v,\omega_v,\mathcal{Q}_v\right)$，且 $\nabla^-_M = \nabla_M\left(\omega^-_N,\mathcal{Q}_n,\Gamma_N\right)$，其中 Γ_N 是克里斯托费尔联络，且

$$\omega^-_{Mab} \equiv \omega_{Mab} - \frac{1}{2}\mathcal{M}_{MN}\mathcal{F}^N_{\mu\nu}e^\mu_a e^\nu_b. \tag{270}$$

For the transformation of all the fields in the theory under the generalized diffeo-morphisms and the external diffeomorphisms, see [246].

关于该理论中所有场在广义微分同胚和外微分同胚下的变换，参见文献 [246]。

$E_{7(7)}$ Exceptional Supergravity in (4 + 56) Dimensions

$E_{7(7)}$ (4+56) 维例外超引力

The bosonic sector was constructed in [247] and the supersymmetric version in [248]. We shall summarize the key results below based on these papers. The field content is given by

玻色子部分由文献 [247] 构造，超对称版本由文献 [248] 完成。下文将基于这两篇论文总结核心结果。场内容由下式给出:

$$\left(g_{\mu\nu},\mathcal{V}_M{}^A,A^M_\mu,B_{\mu\nu\alpha},B_{\mu\nu M};\psi^i_\mu,\chi^{ijk}\right), \tag{271}$$

where $M = 1,\ldots,56$ and $\mathcal{V}_M{}^A = \left(\mathcal{V}_M{}^{ij},\mathcal{V}_{Mij}\right)$ is a representative of the coset $E_{7(7)}/SU(8)$, satisfying $\mathcal{V}_{Mij} = \left(\mathcal{V}_M{}^{ij}\right)^\star$ with $SU(8)$ indices $i,j,\ldots = 1,\ldots,8$. The generalized metric is defined as $\mathcal{M}_{MN} = \mathcal{V}_{Mij}\mathcal{V}_N{}^{ij} + \mathcal{V}_{Nij}\mathcal{V}_M{}^{ij}$. The dual gauge potentials are introduced so that $A^M_\mu = \left(A^m_\mu, A_{\mu m}\right)$ form the 56-plet of $E_{7(7)}$. The two-forms $B_{\mu\nu\alpha}, \alpha = 1,\ldots,133$, are introduced as on-shell duals of the scalars. The two-form potentials $B_{\mu\nu M}$ is introduced as a new type of field which does not follow from the tensor hierarchy. It is needed for achieving a closed generalized gauge algebra. The fermions are Majorana, and $\chi^{ijk} = \chi^{[ijk]}$.

其中 $M = 1, \dots, 56$ 和 $\mathcal{V}_M{}^A = (\mathcal{V}_M{}^{ij}, \mathcal{V}_{Mij})$ 是陪集 $E_{7(7)}/SU(8)$ 的一个代表元，满足带有 $SU(8)$ 指标 $i, j, \dots = 1, \dots, 8$ 的 $\mathcal{V}_{Mij} = (\mathcal{V}_M{}^{ij})^\star$ 。广义度规定义为 $\mathcal{M}_{MN} = \mathcal{V}_{Mij}\mathcal{V}_N{}^{ij} + \mathcal{V}_{Nij}\mathcal{V}_M{}^{ij}$ 。引入对偶规范势使得 $A_\mu^M = (A_\mu^m, A_{\mu m})$ 构成 $E_{7(7)}$ 的 56 重态。引入二形式 $B_{\mu\nu\alpha}, \alpha = 1, \dots, 133$ 作为标量的在壳对偶。二形式势 $B_{\mu\nu M}$ 是一类不遵循张量层次结构的新场，它是获得闭合广义规范代数所必需的。费米子是马约拉纳费米子，即 $\chi^{ijk} = \chi^{[ijk]}$ 。

The covariant two-form field strength is given by

协变二形式场强为:

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}^M - \left[A_\mu, A_\nu\right]_E^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MN}B_{\mu\nu N}. \tag{272}$$

The field strengths $\mathcal{H}_{\mu\nu\rho\alpha}$ and $H_{\mu\nu\rho M}$ can be read off from the Bianchi identity,

场强 $\mathcal{H}_{\mu\nu\rho\alpha}$ 和 $H_{\mu\nu\rho M}$ 可从比安基恒等式得到:

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = -4(t^\alpha)^{MN}\partial_N H_{\mu\nu\rho\alpha} - \frac{1}{6}\Omega^{MN}H_{\mu\nu\rho N},\ D_\mu = \partial_\mu - \mathcal{L}_{A^\mu}.$$

(273)

The three forms arising in the tensor hierarchy, namely $C_{\mu\nu\rho}{}^M{}_\alpha$ and $C_{\mu\nu\rho M}{}^N$, have dropped out by using the section conditions. We also need the scalar current and composite connection defined as [188]

张量层次中出现的三形式，即 $C_{\mu\nu\rho}{}^M{}_\alpha$ 和 $C_{\mu\nu\rho M}{}^N$ ，可以通过利用截面条件消去。我们还需要定义如文献 [188] 的标量流和复合联络:

$$\mathcal{P}_{\mu ijk\ell} = i\Omega^{MN}D_\mu\mathcal{V}_{Mij}\mathcal{V}_{Nk\ell},\ \mathcal{Q}_{\mu i}{}^j = \frac{2i}{3}\mathcal{V}^{Njk}D_\mu\mathcal{V}_{Nki}. \tag{274}$$

Given these building blocks, the (pseudo) Lagrangian can be written as

给定这些构造模块，(赝) 拉氏量可以写为:

$$e^{-1}\mathcal{L} = \widehat{R} + \frac{1}{2}\mathcal{P}_\mu^{ijk\ell}\mathcal{P}^\mu_{ijk\ell} - \frac{1}{8}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N - V(\mathcal{M}, g) + e^{-1}\mathcal{L}_{\text{top}}, \tag{275}$$

with $(\widehat{R}, V)$ from (256) and (257), respectively. The topological Lagrangian, as well as its general variation

其中 $(\widehat{R}, V)$ 分别来自式 (256) 和 (257)。拓扑拉氏量及其全变分

$$\delta\mathcal{L}_{\text{top}} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\left(\delta A_\mu{}^M D_\nu\mathcal{F}_{\rho\sigma M} + \mathcal{F}_{\mu\nu M}\left(6(t^\alpha)^{MN}\partial_N\Delta B_{\rho\sigma\alpha} + \frac{1}{4}\Omega^{MN}\Delta B_{\rho\sigma N}\right)\right),$$

(276)

with the appropriate covariant variations, are given in [247]. It is important to note that this gives a pseudo-action in the sense that it needs to be supplemented by the duality equation

配合合适的协变变分，都已在文献 [247] 中给出。需要注意的是，这给出的是一个赝作用量，需要补充对偶方程

$$\mathcal{F}_{\mu\upsilon}{}^{M} = \frac{i}{2}\varepsilon_{\mu\upsilon\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}\mathcal{F}^{\rho\sigma K}, \tag{277}$$

which must be imposed by hand after performing the general variations in getting the equations of motion. The field strengths $\mathcal{H}_{\mu\upsilon\rho\alpha}$ and $\mathcal{H}_{\mu\upsilon\rho M}$ obey duality equations which can be derived from the variation of the pseudo-action, and the external curl of the duality equation (277). The resulting equations are [247]

在通过全变分得到运动方程后，必须手动加上该对偶方程。场强 $\mathcal{H}_{\mu\upsilon\rho\alpha}$ 和 $\mathcal{H}_{\mu\upsilon\rho M}$ 满足的对偶方程可以从赝作用量的变分以及对偶方程 (277) 的外旋导出，最终得到的方程为 [247]:

$$\varepsilon^{\mu\upsilon\rho\sigma}\mathcal{H}_{\upsilon\rho\sigma\alpha} = -\frac{1}{2}(t_\alpha)_K{}^L\left(D^\mu\mathcal{M}^{KP}\mathcal{M}_{LP}\right),$$

$$\frac{1}{12}\varepsilon^{\mu\upsilon\rho\sigma}\mathcal{H}_{\upsilon\rho\sigma M} = -\frac{1}{3}D^\mu\mathcal{V}^{Nij}\partial_M\mathcal{V}_{Nij} - 2ie^\mu e_b^\upsilon\left(\partial_M\omega_\upsilon{}^{ab} - D_\upsilon\Omega_M{}^{ab}\right),$$

(278)

where $\omega_M{}^{ab} := -e^{\upsilon[a}\partial_M e_\upsilon^{b]}$. The first equation only arises under the projection $(t^\alpha)^{MN}\partial_N$. Finally, the supersymmetry transformations are [248]

其中 $\omega_M{}^{ab} := -e^{\upsilon[a}\partial_M e_\upsilon^{b]}$ 。第一个方程仅在投影 $(t^\alpha)^{MN}\partial_N$ 下存在。最后，超对称变换为 [248]:

$$\delta\psi_\mu^i = 2\mathcal{D}_\mu\varepsilon^i - 4i\mathcal{V}^{Mij}\nabla_M^+\left(\gamma_\mu\varepsilon_j\right),$$

$$\delta\chi^{ijk} = -2\sqrt{2}\mathcal{P}_\mu{}^{ijkl}\gamma^\mu\varepsilon_\ell - 12\sqrt{2}i\mathcal{V}^{M[ij}\nabla_M^+\varepsilon^{k]}, \tag{279}$$

where $\mathcal{D}_\mu = \mathcal{D}_\mu\left(A_\upsilon, \omega_\upsilon, \mathcal{Q}_\upsilon\right)$ and $\nabla_M^+ = \nabla_M\left(\omega_N^+, \mathcal{Q}_n, \Gamma_N\right)$ with Γ_N the Christophel connection and

其中 $\mathcal{D}_\mu = \mathcal{D}_\mu\left(A_\upsilon, \omega_\upsilon, \mathcal{Q}_\upsilon\right)$ 和 $\nabla_M^+ = \nabla_M\left(\omega_N^+, \mathcal{Q}_n, \Gamma_N\right)$，$\Gamma_N$ 为克里斯托费尔联络，且

$$\omega_{Mab}^+ \equiv \omega_{Mab} + \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\upsilon}^N e_a^\mu e_b^\upsilon. \tag{280}$$

In passing, let us also note that a superspace formulation in which the external spacetime is elevated to a (4|32) dimensional superspace, but with the internal space left intact, was constructed in [262].

另外我们也注意到，文献 [262] 构造了一种超空间表述: 将外部时空提升为 (4|32) 维超空间，而内部空间保持不变。

$E_{8(8)}$ Exceptional Supergravity in (3 + 248) Dimensions

$E_{8(8)}$ (3+248) 维例外超引力

The bosonic sector was constructed in [249], and the supersymmetric version in [250]. We will follow these papers for a brief summary here. The field content of the theory is

玻色子部分由文献 [249] 构造，超对称版本由文献 [250] 完成。本文将基于这些文献做简要概述。该理论的场内容为

$$\left(g_{\mu\nu}, \mathcal{V}_M{}^X, A_\mu^M, B_{\mu M}; \psi_\mu^I, \chi^{\dot{A}}\right), \tag{281}$$

where $M, X = 1, \ldots, 248$ and $\mathcal{V}_M{}^X = \left(\mathcal{V}_M{}^{IJ}, \mathcal{V}_M{}^A\right)$ is the representative of the coset $E_{8(8)}/SO(16)$ in the adjoint representation, satisfying $\mathcal{V}_M{}^{IJ} = \mathcal{V}_M{}^{[IJ]}$ with $SO(16)$ vector indices $I, J = 1, \ldots, 16$, and $SO(16)$ spinor indices $A, \dot{A} = 1, \ldots, 128$. The spinors are Majorana. The generalized metric is defined as $\mathcal{M}_{MN} = \mathcal{V}_M{}^X \mathcal{V}_N{}^X$. The potential A_μ^M is dual to the scalar fields, while the one-form potential $B_{\mu M}$ is introduced is a new type of field which does not follow from tensor hierarchy. It is needed for achieving the closure of the generalized gauge algebra, as discussed in section 13.1. See [249] for the details.

其中 $M, X = 1, \ldots, 248$ 和 $\mathcal{V}_M{}^X = \left(\mathcal{V}_M{}^{IJ}, \mathcal{V}_M{}^A\right)$ 是陪集 $E_{8(8)}/SO(16)$ 在伴随表示中的代表元，满足 $\mathcal{V}_M{}^{IJ} = \mathcal{V}_M{}^{[IJ]}$ ，其中 $SO(16)$ 为矢量指标 $I, J = 1, \ldots, 16$ ，$SO(16)$ 为旋量指标 $A, \dot{A} = 1, \ldots, 128$ 。旋量是马约拉纳旋量。广义度规定义为 $\mathcal{M}_{MN} = \mathcal{V}_M{}^X \mathcal{V}_N{}^X$ 。势 A_μ^M 是标量场的对偶，而引入的一元势 $B_{\mu M}$ 是一类新场，并非源自张量分层。正如第 13.1 节讨论的，它是实现广义规范代数封闭所必需的。详见文献 [249]。

The Lagrangian is given by [249]

拉格朗日量由文献 [249] 给出

$$e^{-1}\mathcal{L} = -\widehat{R} + \mathcal{P}_\mu^A \mathcal{P}^{\mu A} + \mathcal{L}_{\text{top}} - V, \tag{282}$$

where $\mathcal{P}_\mu{}^A$ is defined by

其中 $\mathcal{P}_\mu{}^A$ 定义为

$$\mathcal{M}^{KP} D_\mu \mathcal{M}_{PL} = 2 f^{MK}{}_L \mathcal{V}_M{}^A \mathcal{P}_\mu{}^A, \ D_\mu = \partial_\mu - \mathcal{L}_{(A_\mu, B_\mu)}. \tag{283}$$

The topological action is given by [249]

拓扑作用量由文献 [249] 给出

$$S_{\text{top}} = \int d^{248} Y \int_{\mathcal{M}_4} \left(\mathcal{F}^M \wedge \mathcal{G}_M - \frac{1}{2} f_{MN}{}^K \mathcal{F}^M \wedge \partial_K \mathcal{F}^N\right), \tag{284}$$

with the three-dimensional spacetime residing on boundary of $\mathcal{M}_4$. The field strengths are given by

其中三维时空位于 $\mathcal{M}_4$ 的边界上。场强由下式给出

$$\mathcal{F}_{\mu\nu}{}^M = F_{\mu\nu}{}^M + 14 (P_{3875})^{MN}{}_{PQ} \partial_N C_{\mu\nu}{}^{PQ} + \frac{1}{4} \partial^M C_{\mu\nu} + 2 f^{MN}{}_P C_{\mu\nu N}{}^P,$$

$$\mathcal{G}_{\mu\nu M} = G_{\mu\nu M} + 2\partial_N C_{\mu\nu M}{}^N + 2\partial_M C_{\mu\nu N}{}^N, \tag{285}$$

where $F_{\mu\nu}{}^M$ and $G_{\mu\nu M}$ are determined from $\left[D_\mu, D_\nu\right] V^M = -\mathcal{L}_{(F_{\mu\nu}, G_{\mu\nu})} V^M$ and given explicitly in [249], where it is also shown that all dependence on the two-form fields drop out of the action. The physical degree of freedoms are in accordance with supersymmetry, in view of the fact that the general covariant variation of the action with respect to field $B_{\mu M}$ gives the duality equation

其中 $F_{\mu\nu}{}^M$ 和 $G_{\mu\nu M}$ 由 $\left[D_\mu, D_\nu\right] V^M = -\mathcal{L}_{(F_{\mu\nu}, G_{\mu\nu})} V^M$ 确定，并在文献 [249] 中有明确表达式，该文献同时证明作用量中所有对二元场的依赖都会消去。考虑到作用量对场 $B_{\mu M}$ 的广义协变变分给出对偶方程

$$\mathcal{F}_{\mu\nu}{}^M = \frac{1}{60} \varepsilon_{\mu\nu\rho} f^{MK}{}_L \left(\mathcal{M}^{LP} D^\rho \mathcal{M}_{PK}\right), \tag{286}$$

which holds up to terms of the form $\mathcal{O}_{\mu\nu}{}^M$ that vanish when contracted with a field satisfying the section constraints, since the general variation is with respect to $B_{\mu M}$ which is subject to the section constraints. Finally, the supertransformation rules are [249]

该方程在相差 $\mathcal{O}_{\mu\nu}{}^M$ 形式的项范围内成立，这类项与满足截面约束的场缩并时会消失，因为广义变分是对服从截面约束的 $B_{\mu M}$ 做的。最后，超变换规则为 [249]

$$\delta\psi^I_\mu = \mathcal{D}_\mu \varepsilon^i + 2i\mathcal{V}^M{}_{IJ} \nabla_M \left(\gamma_\mu \varepsilon^J\right) + 2i\mathcal{V}^M{}_{IJ} \gamma_\mu \nabla_M \varepsilon^J,$$

$$\delta\chi^{\dot{A}} = \frac{i}{2} \gamma^\mu \varepsilon^i \Gamma^I_{A\dot{A}} \mathcal{P}^A_\mu - 2\mathcal{V}^M{}_A \Gamma^I_{A\dot{A}} \nabla_M \varepsilon_I \tag{287}$$

$N = 1$ Supersymmetric Double Field Theory

$N = 1$ 超对称双重场论

$N = 1, 10D$ supergravity coupled to an arbitrary number of abelian vector multiplets was constructed in [255], and its generalization to non-abelian coupling of n_V vector fields in [263]. The main results can be summarized briefly as follows. The extended geometrical framework in this case is based on the coset $G/H = O(10, 10 + n_V) / (O(9,1) \times O(1, 9 + n_V))$. The multiplet of fields consists of

$N = 1, 10D$ 超引力耦合任意数目的阿贝尔矢量多重态的理论已在文献 [255] 中构建，其推广到 n_V 矢量场的非阿贝尔耦合的工作见文献 [263]。主要结果可简要总结如下。这类情形下的扩展几何框架基于陪集 $G/H = O(10, 10 + n_V) / (O(9,1) \times O(1, 9 + n_V))$. The multiplet of fields consists of

$$\left(E^M A, d, \Psi_{\bar{a}}, \rho\right), \tag{288}$$

where $E^M{}_A = \left(E^M{}_a, E^M{}_{\bar{a}}\right), a = 0, 1, \ldots, 9, \bar{a} = 0, 1, \ldots, 9 + n_V$, is the generalized vielbein and d is the dilaton. The gravitino $\Psi_{\bar{a}}$ and dilatino ρ are Majorana spinors of $O(9,1)$, and the gravitino is a vector of $O(1, 9 + n_V)$. The bosonic part of the Lagrangian is given by $e^{-2d} R$ where R is the generalized Ricci scalar, thus taking the form [255, 263]

其中 $E^M{}_A = \left(E^M{}_a, E^M{}_{\bar{a}}\right), a = 0, 1, \ldots, 9, \bar{a} = 0, 1, \ldots, 9 + n_V$ 是广义标架，d 是 dilaton(dilaton)，引力微子 $\Psi_{\bar{a}}$ 和伸缩微子 ρ 是 $O(9,1)$ 的马约拉纳旋量，且引力微子是 $O(1, 9+n_V)$ 的矢量。拉氏量的玻色部分由 $e^{-2d}R$ 给出，其中 R 是广义里奇标量，因此形式为 [255, 263]

$$\mathcal{L} = e^{-2d}\left[\frac{1}{8}\omega_{[ABC]}\omega_{[DEF]}\left(H^{AD}\eta^{BE}\eta^{CF} - \frac{1}{3}H^{AD}H^{BE}H^{CF}\right)\right.$$

$$\left. - H^{AB}\left(E_A + \frac{1}{2}\omega_A\right)\omega_B\right], \tag{289}$$

where $\omega_A = \omega_{BA}{}^B, E_A = -\sqrt{2}E_A{}^M\partial_M$, and the totally antisymmetric part of the spin connection

其中 $\omega_A = \omega_{BA}{}^B, E_A = -\sqrt{2}E_A{}^M\partial_M$ ，以及自旋联络的全反对称部分

$$\omega_{[ABC]} = \left[\left(E_A E^N{}_B\right)E_{NC} - \frac{\sqrt{2}}{3}f_{MNP}E^M{}_A E^N{}_B E^P{}_C\right]_{[ABC]}. \tag{290}$$

The symmetric and invertible constant metrics η_{AB} and H_{AB} are H -invariant metrics, with the latter constrained to satisfy $H_A{}^C H_C{}^B = \delta_A^B$. The constant metric η_{MN} is G invariant. The metrics η_{AB} and η_{MN} are used to raise and lower indices. The supertransformations of the fermionic fields are given by

对称且可逆的常数度规 η_{AB} 和 H_{AB} 是 H 不变度规，其中后者受约束满足 $H_A{}^C H_C{}^B = \delta_A^B$ 。常数度规 η_{MN} 是 G 不变的。度规 η_{AB} 和 η_{MN} 用于升降指标。费米场的超变换由下式给出

$$\delta\Psi_{\bar{a}} = \nabla_{\bar{a}}\varepsilon, \ \delta\rho = -\gamma^a\nabla_a\varepsilon, \tag{291}$$

where ε is a Majorana spinor of $O(1,9)$, and $\nabla_A\varepsilon = \left(E_A - \frac{1}{4}\omega_{Abc}\gamma^{bc}\right)\varepsilon$.

其中 ε 是 $O(1,9)$ 的马约拉纳旋量，且 $\nabla_A\varepsilon = \left(E_A - \frac{1}{4}\omega_{Abc}\gamma^{bc}\right)\varepsilon$ 。

Consistent Kaluza-Klein Reductions

一致卡鲁扎-克莱因约化

Generalized Scherk-Schwarz ansatz for the full supersymmetric $E_{6(6)}$ and $E_{7(7)}$ exceptional field theories that uses twist matrices subject to consistency equations, and leading to the field equations of lower dimensional gauged supergravity theories parametrized by an embedding tensor, was achieved in [256]. Following [256], let us briefly summarize how this works for $E_{6(6)}$. Under $SL(6)\times SL(2)$ the coordinates in 27 of $E_{6(6)}$ are decomposed as $Y^M \to \left(Y^{[AB]}, Y_{A\alpha}\right)$ with $A = 0, 1, \ldots, 5, \alpha = 1{,}2$, and all fields are taken to depend only on $\left(x^\mu, Y^{0i}\right)$ with $\mu = 0, 1, \ldots, 5$ and $i = 1, \ldots, 5$. Next, in a Scherk-Schwarz reduction scheme, the $E_{6(6)}$ twist matrix is chosen as [256]

针对全超对称 $E_{6(6)}$ 与 $E_{7(7)}$ 例外场论，文献 [256] 实现了广义谢尔克-施瓦茨 ansatz: 该方法使用满足相容性方程的扭曲矩阵,最终得到由嵌入张量参数化的低维定域超引力场方程。遵循 [256] 的内容,我们简要总结该方法对 $E_{6(6)}$ 的过程。在 $SL(6)\times SL(2)$ 下, $E_{6(6)}$ 的 27 维坐标分解为 $Y^M \to \left(Y^{[AB]}, Y_{A\alpha}\right)$，其中 $A=0,1,\dots,5, \alpha=$ 取 1、2，所有场都仅依赖于 $\left(x^\mu, Y^{0i}\right)$，满足 $\mu=0,1,\dots,5$ 和 $i=1,\dots,5$。接下来，在谢尔克-施瓦茨约化框架中，$E_{6(6)}$ 扭曲矩阵取为 [256]

$$U_M{}^N = \begin{pmatrix} U_{[AB]}{}^{[CD]} & 0 \\ 0 & \delta^\alpha_\gamma \left(U^{-1}\right)_C{}^A \end{pmatrix}, \tag{292}$$

and the following ansatz is made for all the fields

并对所有场给出如下 ansatz

$$e_\mu{}^a(x,Y) = \rho^{-1}(Y)\, e_\mu{}^a(x),$$

$$\mathcal{M}_{MN}(x,Y) = U_M{}^P(Y)\, U_N{}^Q(Y)\, M_{PQ}(x),$$

$$A_\mu{}^M(x,Y) = A_\mu{}^N(x)\left(U^{-1}\right)_N{}^M(Y)\,\rho^{-1}(Y),$$

$$B_{\mu\upsilon M}(x,Y) = \rho^{-2}(Y)\, U_M{}^P(Y)\, B_{\mu\upsilon P}(x). \tag{293}$$

The consistency of this reduction is shown to impose the conditions

可证明该约化的相容性要求满足以下条件

$$\left[\left(U^{-1}\right)_M{}^P\left(U^{-1}\right)_N{}^Q \partial_P U_Q{}^K\right]_{(351)} = \rho\theta_M{}^\alpha (t_\alpha)_N{}^K,$$

$$\left[\left(\partial_N - 4\left(\rho^{-1}\partial_N\rho\right)\right]\left(U^{-1}\right)_M{}^N = 3\rho\vartheta, \tag{294}$$

where (351) denotes projection to the 351-dimensional representation of $E_{6(6)}$, and $(\theta_M{}^\alpha, \vartheta_M)$ are the embedding tensor components, the latter associated with the gauging of the trombone symmetry. The solutions of these equations, together with the dictionary that relates exceptional field theories to type IIB supergravity [245], provides consistent truncations to $AdS_5\times S^5$ and various hyperboloid compactifications as well. For more details, see [256].

其中 (351) 表示投影到 $E_{6(6)}$ 的 351 维表示，$(\theta_M{}^\alpha, \vartheta_M)$ 是嵌入张量分量，后者对应长号对称性的定域化。这些方程的解，结合例外场论与 IIB 型超引力的对应字典 [245]，可以给出到 $AdS_5\times S^5$ 的一致截断，也适用于各类双曲面紧致化。更多细节参见 [256]。

In the case of DFTs, by appropriately choosing the global and local symmetry groups and gaugings, one can describe both the $N=1, 10D$ heterotic supergravity as well as its toroidal compactified version. To this end, following the presentation of [89], the groups G and H discussed in section "Extended Geometry" are split as

对于双场论 (DFT)，通过适当选择整体和定域对称性群与定域化，可以同时描述 $N=1,10D$ 杂化超引力及其环面紧致化版本。为此，遵循文献 [89] 的表述，“扩展几何”一节讨论的群 G 和 H 分解为

$$G \to \underbrace{O(d,d)}_{G_e} \times \underbrace{O(n,n+n_V)}_{G_i}$$

$$H \to \underbrace{O(d-1,1)\times O(1,d-1)}_{H_e} \times \underbrace{O(n)\times O(n+n_V)}_{H_i}, \tag{295}$$

and the matrices η_{AB}, H_{AB} and η_{MN} are taken to be

且矩阵 η_{AB}, H_{AB} 和 η_{MN} 取为

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a_b & 0 \\ \delta_a{}^b & 0 & 0 \\ 0 & 0 & \kappa_{mn} \end{pmatrix}, \; H_{AB} = \begin{pmatrix} g^{ab} & 0 & 0 \\ 0 & g_{ab} & 0 \\ 0 & 0 & M_{\alpha\beta} \end{pmatrix}, \; \eta_{MN} = \begin{pmatrix} 0 & \delta^\mu{}_\nu & 0 \\ \delta_\mu{}^\nu & 0 & 0 \\ 0 & 0 & \kappa_{mn} \end{pmatrix},$$

(296)

where $\mu, a = 0, \ldots, D-1, m, \alpha = 1, \ldots, 2n+n_V$. Here g_{ab} is the flat Minkowski metric $\kappa_{\alpha\beta}$ and $M_{\alpha\beta}$ are two symmetric and constant H_i invariant matrices, and κ_{mn} is the G_i invariant metric. A suitable solution to the constraints (261) for the reduction to $D = 10-n$ dimensions is given by

其中 $\mu, a = 0, \ldots, D-1, m, \alpha = 1, \ldots, 2n+n_V$ 。此处 g_{ab} 是平坦闵可夫斯基度规 $\kappa_{\alpha\beta}$ ，$M_{\alpha\beta}$ 是两个对称且常数的 H_i 不变矩阵，κ_{mn} 是 G_i 不变度规。对于约化到 $D=10-n$ 维的情况，约束条件 (261) 的一个合适解为

$$f_{MNP} = \delta^m_M \delta^n_N \delta^p_P f_{mnp}, \; \partial_M = \left(\tilde{\partial}^\mu, \partial_\mu, \partial_m\right) = \left(0, \partial_\mu, 0\right), \tag{297}$$

and the generalized vielbein is parametrized as [89]

广义标架参数化为 [89]

$$E_M{}^A = \begin{pmatrix} e_a{}^\mu & 0 & 0 \\ -e_a{}^\rho \left(B_{\mu\nu} + \frac{1}{2} A_\mu{}^m A_{\nu m}\right) & e_\mu{}^a & A_\mu{}^p \mathcal{V}_p{}^\alpha \\ -e_a{}^\rho A_{\rho m} & 0 & \mathcal{V}_m{}^\alpha \end{pmatrix}, \; e^{-2d} = \sqrt{-g} e^{-2\phi}.$$

(298)

The external double-Lorentz transformations are gauge fixed to the diagonal part corresponding to the single Lorentz group in D dimensions, $\mathcal{V}_m{}^\alpha$ is the vielbein on the scalar manifold G_i/H_i , and $A_\mu{}^m$ is the Yang-Mills gauge field. After some calculation that also involves field redefinitions, the following action is found [89]

外双洛伦兹变换被规范固定到对应 D 维单洛伦兹群的对角部分，$\mathcal{V}_m{}^\alpha$ 是标量流形 G_i/H_i 上的标架，$A_\mu{}^m$ 是杨-米尔斯规范场。经过包含场重定义在内的若干计算，得到如下作用量 [89]

$$S = \int d^D x \sqrt{-g} e^{-2\phi} \Big[R + 4\nabla_\mu \partial^\mu \phi - 4\partial_\mu \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$- \frac{1}{4} M_{mn} F^m_{\mu\nu} F^{\mu\nu n} + \frac{1}{8} \nabla_\mu M_{mn} \nabla^\mu M^{mn} - V \Big], \tag{299}$$

where $M_{mn} = M_{\alpha\beta} \mathcal{V}_m{}^\alpha \mathcal{V}_n{}^\beta$ and

其中 $M_{mn} = M_{\alpha\beta} \mathcal{V}_m{}^\alpha \mathcal{V}_n{}^\beta$ 且

$$H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} - 3\left(A_{[\mu}{}^m \partial v A_{\rho]m} - \frac{1}{3} f_{mnp} A_\mu{}^m A_\nu{}^n A_\rho{}^p \right),$$

$$F_{\mu v}{}^m = 2\partial_{[\mu} A_{v]}{}^m - f_{pq}{}^m A_\mu{}^p A_v{}^q,$$

$$\nabla_\mu M_{mn} = \partial_\mu M_{mn} + f_{pm}{}^q A_\mu{}^p M_{qn} + f_{pn}{}^q A_\mu{}^p M_{mq}, \tag{300}$$

and the potential is given by

势由下式给出

$$V = \frac{1}{12} f_{mp}{}^r f_{nq}{}^s M^{mn} M^{pq} M_{rs} + \frac{1}{4} f_{mp}{}^q f_{nq}{}^p M^{mn} + \frac{1}{6} f_{mnp} f^{mnp}. \tag{301}$$

The constants f_{mnp} break the global duality symmetry $O(n, n+n_V)$, but the action still provides an $O(n, n+n_V)$ covariant description. In fact the action above has the same form as the half-maximal gauged supergravities reviewed in previous sections. For example, taking $n = 3$, we have checked that the potential (301) agrees with the one given in (63), which was obtained directly in $7D$ by means of Noether procedure.

常数 f_{mnp} 破缺了整体对偶对称性 $O(n, n+n_V)$ ，但该作用量仍给出了一个 $O(n, n+n_V)$ 协变描述。事实上上述作用量与前文回顾的半极大规范超引力形式一致。例如，取 $n = 3$ ，我们验证了势 (301) 与 (63) 中给出的形式一致，后者是通过诺特定 procedure 在 $7D$ 中直接得到的。

The reduction of $N = 1, 10D$ heterotic supergravity with Yang-Mills symmetry group K was performed in [101], leading to a large class of gauged supergravities in $d = 10 - n$ dimensions, where $n = \dim \mathrm{G}$. The hidden $O(n, n + \dim \mathrm{K})$ symmetry was uncovered, and local gauge symmetry in d dimensions as $K \times G \rtimes R^n$ was identified. An alternative route to gauged supergravities in lower dimensions is to perform generalized Scherk-Schwarz reduction in the framework of the DFT. This has been done in [194, 264] where $N = 4, 4D$ supergravity with its electric gaugings were obtained in agreement with the embedding tensor construction of [82]. A fuller analysis awaits to be performed to pin down the most general gaugings, and their relationships to higher dimensions.

带有杨-米尔斯对称群 K 的 $N = 1, 10D$ 杂化超引力约化已在文献 [101] 中完成，得到了一大类 $d = 10 - n$ 维规范超引力，其中 $n = \dim \mathrm{G}$ 。该工作揭示了隐藏的 $O(n, n + \dim \mathrm{K})$ 对称性，并确定了 d 维下作为 $K \times G \rtimes R^n$ 的局域规范对称性。得到低维规范超引力的另一途径是在双场理论框架下进行广义舍尔克-施瓦茨约化，这一工作已在 [194, 264] 中完成，其中得到了带电力规范的 $N = 4, 4D$ 超引力，与文献 [82] 的嵌入张量构造一致。要确定最一般的规范及其与高维理论的关系，仍有待更全面的分析。

A Conventions, and Spinors in Arbitrary Dimensions

A 约定与任意维度中的旋量

Throughout the paper, it is understood that the supersymmetry transformation rules are given up to terms that are quadratic in the fermionic fields of the supermultiplets involved. As for the signature of spacetime, and conventions, we mostly stick to the conventions of the original papers, though some notation changes are employed in certain cases. Nonetheless, it is useful to recall the general properties of spinors and Dirac gamma-matrices in arbitrary dimensions, which we reproduce here for the reader's convenience. This also gives us an opportunity to correct the typos that appeared in equations (3), (6), (8), and (9) on pages 5 and 6 in [3].

全文默认，超对称变换规则仅保留到所涉及超多重态费米场的二次项。关于时空号差与整体约定，我们大多遵循原文献的设定，仅在部分场景调整了部分记号。尽管如此，梳理任意维度下旋量与狄拉克伽马矩阵的一般性质仍有必要，我们在此整理出来供读者参考，同时也借此机会修正文献 [3] 第 5、6 页中方程 (3)、(6)、(8)、(9) 的笔误。

Consider the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ in (t, s) dimensions with the signature

考虑 $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ 维空间中号差为的克利福德代数 (t, s)

$$\eta_{\mu\nu} = \text{diag}\left(\underbrace{-, -, \ldots -}_{t \text{ times}}, \underbrace{+, +, \ldots +}_{p \text{ times}}\right). \tag{302}$$

The γ -matrices have the following properties

γ 矩阵具有如下性质

$$\gamma_\mu^\dagger = (-1)^t A\gamma_\mu A^{-1},\ \gamma_\mu^\star = \eta B\gamma_\mu B^{-1},\ \gamma_\mu^T = (-1)^t \eta C\gamma_\mu C^{-1},$$

(303)
where

其中

$$A = \gamma_0\gamma_1 \ldots \gamma_{t-1},\ B^T = \varepsilon B,\ C = BA,\ \varepsilon = \pm 1,\ \eta = \pm 1.$$

(304)
It follows that

由此可得

$$\left(C\gamma_{\mu_1\mu_2\ldots\mu_n}\right)^T = \varepsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}\left(C\gamma_{\mu_1\mu_2\ldots\mu_n}\right). \tag{305}$$

Same symmetry property applies to $\gamma_{\mu_1\mu_2\ldots\mu_n}C^{-1}$. Note also that $C\gamma_{\mu_1\mu_2\ldots\mu_n}$ and $C\gamma_{\mu_1\mu_2\ldots\mu_{n+2}}$ have the opposite symmetry property. The Fierz rearrangement formula for two $2^{[d/2]}$ dimensional anti-commuting spinors ψ and χ in d -dimensions is given by

相同的对称性也适用于 $\gamma_{\mu_1\mu_2\ldots\mu_n}C^{-1}$ 。还需注意 $C\gamma_{\mu_1\mu_2\ldots\mu_n}$ 与 $C\gamma_{\mu_1\mu_2\ldots\mu_{n+2}}$ 的对称性相反。对于 d 维空间中两个 $2^{[d/2]}$ 维的反对易旋量 ψ 和 χ ，菲尔茨重排公式为

$$\chi\bar{\psi} = -2^{-[d/2]}\sum_{n=0}^{d}(-1)^{[n/2]}\gamma^{v_1\ldots v_n}\left(\bar{\psi}\gamma_{v_1\ldots v_n}\chi\right)/n! \tag{306}$$

The product of n -th rank and m -th γ -matrices can be decomposed as

n 秩与 m 秩 γ 矩阵的乘积可分解为

$$\gamma_{\mu_1\ldots\mu_m}\gamma^{v_1\ldots v_n} = \gamma_{\mu_1\ldots\mu_m}{}^{v_1\ldots v_n} + mn\delta^{v_1}_{\mu_m}\gamma_{\mu_1\ldots\mu_{m-1}}{}^{v_2\ldots v_n}$$

$$+\frac{1}{2!}m(m-1)\,n(n-1)\,\delta^{v_1}_{\mu_m}\delta^{v_2}_{\mu_{m-1}}\gamma_{\mu_1\ldots\mu_{m-2}}{}^{v_3\ldots v_n} + \ldots, \tag{307}$$

where it is understood that the indices $\mu_1\ldots\mu_m$ and $v_1\ldots v_n$ are to be totally anti-symmetrized with unit normalization. Finally, the reality properties of spinors in (t,s) dimensions are tabulated below, where $\Omega^{ij} = -\Omega^{ji}$ is invariant tensor of the symplectic group. Additional Weyl condition can be imposed for $(t-s) = 0 \mod 4$.

其中 $\mu_1\ldots\mu_m$ 与 $v_1\ldots v_n$ 指标需按单位归一化做全反对称化。最后，(t,s) 维中旋量的实性性质整理如下表，其中 $\Omega^{ij} = -\Omega^{ji}$ 是辛群的不变张量。对于 $(t-s) = 0 \mod 4$ 还可额外附加外尔条件。

$(s-t)$	ε	η	Reality Condition	Spinor
1,2,8	+1	+1	$\psi^* = B\psi$	Majorana
6,7,8	+1	-1	$\psi^* = B\psi$	Pseudo-Majorana
4,5,6	-1	+1	$(\psi_i)^* = \varnothing^{ij}B\psi_j$	Sympl. Majorana
2,3,4	-1	-1	$(\psi_i)^* = \varnothing^{ij}B\psi_j$	Pseudo-Sympl-Majorana

B The Embedding Tensor Formalism

B 嵌入张量形式论

For a detailed account of the embedding tensor formalism, sometimes referred to as the tensor hierarchy formalism, and for more complete references, see [265, 266]. Here, we shall primarily follow [266] to give a brief outline.

关于有时也被称为张量层级形式论的嵌入张量形式论，如需详细说明及更完整参考文献，参见 [265, 266]。本文此处主要遵循文献 [266] 给出简要概述。

Consider a Lie group G_0 with associated Lie algebra $\left[t_\alpha, t_\beta\right] = f_{\alpha\beta}{}^{\gamma}t_\gamma$. Next, suppose that we wish to gauge a subgroup $G \subset G_0$ with generators X_M given as a linear combination of generators t_α as

考虑一个李群 G_0 ，其对应李代数为 $\left[t_\alpha, t_\beta\right] = f_{\alpha\beta}{}^\gamma t_\gamma$ 。接下来假设我们要规范一个子群 $G \subset G_0$ ，其生成元 X_M 可表示为生成元 t_α 的线性组合，形式如下

$$X_M = \theta_M{}^\alpha t_\alpha,\ M = 1, \ldots, \dim R_1,\ \alpha = 1, \ldots, \dim G_0, \tag{308}$$

where R_1 is a representation of group G , and $\theta_M{}^\alpha$ is a constant matrix, called the embedding tensor. We demand that X_M satisfy a Lie algebra

其中 R_1 是群 G 的一个表示，$\theta_M{}^\alpha$ 是一个常数矩阵，称为嵌入张量。我们要求 X_M 满足李代数

$$[X_M, X_N] = X_{MN}{}^P X_P, \tag{309}$$

with $X_{MN}{}^P$ representing the structure constants. This imposes the condition

其中 $X_{MN}{}^P$ 代表结构常数。这给出了条件

$$\theta_M{}^\alpha \theta_N{}^\beta f_{\alpha\beta}{}^\gamma + \theta_M{}^\alpha (t_\alpha)_N{}^P \theta_P{}^\gamma = 0. \tag{310}$$

This, in turn, ensures that the embedding tensor is G -invariant. Note that $X_{MN}P$ need not be antisymmetric, and indeed it is useful to define

这反过来又能保证嵌入张量是 G 不变的。注意 $X_{MN}P$ 不必是反对称的，因此我们可以定义

$$X_{(MN)}{}^P \ := \ Z^P{}_{MN}. \tag{311}$$

Thus, (309) implies that

因此，式 (309) 给出

$$\theta_P{}^\alpha Z^P{}_{MN} = 0. \tag{312}$$

Next, one seeks a proper definition of gauge transformation of the gauge field, and a field strength that transforms covariantly under the gauge group G . This innocent demand turns out to require the introduction of a hierarchy of p -forms, gauge transformations involving shift symmetries, and constant intertwining tensors that have following pattern

接下来，我们需要给出规范场规范变换的恰当定义，以及一个在规范群 G 下协变变换的场强。这个看似简单的要求实际上需要引入一套 p -形式的层级结构、包含平移对称性的规范变换，以及满足下述模式的常数缠绕张量:

$$p\text{ - forms}:\ A_\mu{}^M \to B_{\mu\nu}{}^{MN} \to C_{\mu\nu\rho}{}^{MNP} \to \cdots$$

$$\text{Gauge parameters}:\ \Lambda^M \to \textstyle\sum_\mu{}^{MN} \to \Phi_{\mu\nu}{}^{MNP} \to \cdots$$

规范参数: $\Lambda^M \to \sum_\mu{}^{MN} \to \Phi_{\mu\nu}{}^{MNP} \to \cdots$

$$\text{Intertwiners}: Z^M{}_{NP} \to Y^{NP}{}_{QRS} \to Y^{QRS}{}_{TUVW} \to \cdots$$

(313)

The gauge transformations are

规范变换形式为

$$\delta A_\mu{}^M = D_\mu \Lambda^M - Z^M{}_{NP} \sum_\mu{}^{NP},$$

$$\delta B_{\mu\nu}{}^{MN} = 2D_{[\mu} \sum_{\nu]}{}^{MN} - 2\Lambda^{\langle M} \mathcal{F}_{\mu\nu}^{N\rangle} - Y^{MN}{}_{P\langle RS\rangle} \Phi_{\mu\nu}{}^{PRS} + 2A_{[\mu}^{\langle M} \delta A_{\nu]}^{N\rangle},$$

$$\vdots \qquad (314)$$

where $D_\mu \Lambda^M = \partial_M \Lambda^M + X_{KL}{}^M A_\mu^K \Lambda^L$, and the notation $\langle MN \rangle$ refers to particular representation obtained by applying the corresponding projector $\mathbb{P}^{KL}{}_{MN}$ on the symmetric product $R_1 \otimes_{\text{sym}} R_1$ as follows

其中 $D_\mu \Lambda^M = \partial_M \Lambda^M + X_{KL}{}^M A_\mu^K \Lambda^L$ ，记号 $\langle MN \rangle$ 指的是对对称乘积 $R_1 \otimes_{\text{sym}} R_1$ 施加对应投影算符 $\mathbb{P}^{KL}{}_{MN}$ 后得到的特定表示，如下所示

$$Z^P{}_{\langle MN \rangle} = Z^P{}_{KL} \mathbb{P}^{KL}{}_{MN}. \qquad (315)$$

The intertwining Y -tensors are linear in the embedding tensor and they are the maps

缠绕 Y 张量是嵌入张量的线性函数，它们是如下映射:

$$Y^{[p]} : R_1^{\otimes(p+1)} \to R_1^{\otimes p}, \qquad (316)$$

with $\left(Y^{[0]}\right)^\alpha{}_M = \theta_M{}^\alpha$ and $\left(Y^{[1]}\right)^M{}_{PQ} = Z^M{}_{PQ}$, etc., with the property that

其中 $\left(Y^{[0]}\right)^\alpha{}_M = \theta_M{}^\alpha$ 、$\left(Y^{[1]}\right)^M{}_{PQ} = Z^M{}_{PQ}$ 以此类推，具有如下性质

$$Y^{[p]} \cdot Y^{[p+1]} \approx 0, \qquad (317)$$

where "?" means that the expression vanishes a consequence of (310). This map has a non-trivial kernel whose complement defines the representation content of $(p+1)$ -forms required for the consistency of deformed p -form gauge algebra.

其中"?" 表示该表达式作为式 (310) 的结果等于零。该映射具有非平凡核，其补空间定义了保证形变后的 p -形式规范代数自洽所需的 $(p+1)$ -形式表示内容。

The G -covariant field strengths are constructed as

G 协变场强构造如下:

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}A^{M}_{\nu]} + X_{KL}{}^{M}A^{K}_{[\mu}A^{L}_{\nu]} + Z^{M}{}_{KL}B_{\mu\nu}{}^{KL},$$

$$\mathcal{H}_{\mu\nu\rho}{}^{MN} = 3D_{[\mu}B_{\nu\rho]}{}^{MN} + 6A^{\langle M}_{[\mu}\partial_{\nu}A^{N\rangle}_{\rho]} + 2A^{\langle M}_{\mu}X_{PQ}{}^{N\rangle}A^{P}_{\nu}A^{Q}_{\rho}\,|_{[\mu\nu\rho]}$$

$$+Y^{MN}{}_{P\langle RS\rangle}C^{PRS}_{\mu\nu\rho},$$

(318)

The hierarchy can be truncated at a given level if a p -form of maximum rank appears in the Lagrangian in such a way that it is projected by a suitable intertwining tensor. For example, to stop the hierarchy at B^{MN}_{μ} , this two-form can only appear in the combination $Z^{P}{}_{MN}B^{MN}_{\mu\nu}$, and the terms involving $Y^{MN}{}_{PQR}$ will not appear since $Z^{P}{}_{MN}Y^{MN}{}_{PQR} = 0$. For a detailed study of the tensor hierarchy in $D = 5, 6$ including the construction of all the field strengths, see [267].

如果拉格朗日中出现的最大秩 p -形式可被恰当的缠绕张量投影，则该层级可在给定阶截断。例如，若要将层级截断在 B^{MN}_{μ} 阶，该二形式只能以组合 $Z^{P}{}_{MN}B^{MN}_{\mu\nu}$ 的形式出现，涉及 $Y^{MN}{}_{PQR}$ 的项由于 $Z^{P}{}_{MN}Y^{MN}{}_{PQR} = 0$ 不会出现。关于 $D = 5, 6$ 中张量层级的详细研究，包括所有场强的构造，参见 [267]。

In supergravity theories, the requirement of supersymmetry imposes an additional constraint which is linear in the embedding tensor. This plays a crucial role in determining the representation content of the embedding tensor. Moreover, in order to have the correct physical degree of freedom count in a given supermultiplet, the representation content of the p -forms in the tensor hierarchy must be chosen in such a way that a p -form potential in a given representation should come with its dually related $(D - p - 2)$ -form potential in the conjugate representation. The embedding tensor subjected to the linear and quadratic constraints will determine the form of other constant tensors in the hierarchy such as $Z^{P}{}_{MN}, Y^{NP}{}_{QRS}$, depending on the dimension of spacetime, the amount for supersymmetry, and the supermultiplets involved. In sections "D = 9", "D = 8", "D = 7", "D = 6", "D = 5", " D = 4 ", and " D = 3 ", we have summarized how this works explicitly.

在超引力理论中，超对称性要求施加了一个额外约束，该约束对嵌入张量是线性的。这一点在确定嵌入张量的表示内容方面起着至关重要的作用。此外，为了使给定超多重态拥有正确的物理自由度计数，张量层级中 p -形式的表示内容必须按如下方式选取: 给定表示中的一个 p -形式势，其对偶相关的 $(D - p - 2)$ -形式势必须出现在共轭表示中。受线性和二次约束的嵌入张量将决定层级中其他常数张量 (例如 $Z^{P}{}_{MN}, Y^{NP}{}_{QRS}$) 的形式，具体取决于时空维度、超对称性的量以及涉及的超多重态。我们在“D=9”“D=8”“D=7”“D=6”“D=5”“D = 4 ”和“D = 3 ”小节中，明确总结了该机制的具体运作方式。

C Coset Spaces in Supergravities

C 超引力中的陪集空间

Table 4 Scalar manifolds from n_V vector and n_T tensor multiplet couplings for supergravities in $D \geq 6$ dimensions with N supersymmetry. n_V^{tot} is the total number of vector fields, and n_V is the total number of vector multiplets. The last four entries in $6D$ arise in magical supergravities discussed section " $(1,0)$ Magical Supergravities in $6D$ ", and their consecutive circle reductions yield the last four entries for $D = 5, N = 2$ and $D = 4, N = 2$ in Table 5, and for $D = 3, N = 4$ in Table 6. For "twin" supergravities see sections "Comments on Other Supergravities in $6D$ " and " $N = 6$ Supergravity in $5D$ "

表 4 $D \geq 6$ 维空间中具有 N 超对称的超引力，由 n_V 向量多重态和 n_T 张量多重态耦合得到的标量流形。n_V^{tot} 为向量场总数，n_V 为向量多重态总数。$6D$ 中最后四个条目出现在“$(1,0)$ $6D$ 维魔法超引力”一节讨论的魔法超引力中，它们的连续圆约化给出表 5 中 $D = 5, N = 2$ 和 $D = 4, N = 2$ 、表 6 中 $D = 3, N = 4$ 的最后四个条目。关于“孪生”超引力参见“关于 $6D$ 维其他超引力的评论”和“$N = 6$ $5D$ 维超引力”两节

D	N	Scalar Manifold G/H	n_V^{tot}	n_T	Comment
10	(1,1)	$\mathbb{R}$	-	-	
	(2,0)	$SU(1,1)/U(1)$	-	-	
9	2	$\mathbb{R} \times SL(2,\mathbb{R})/SO(2)$	3	-	
	1	$\mathbb{R} \times SO(n_V,1)/SO(n_V)$	$n_V + 1$	-	
8	2	$(SL(3,\mathbb{R})/SO(3)) \times (SL(2,\mathbb{R})/SO(2))$	$3+3$	-	
	1	$\mathbb{R} \times SO(n_V,2)/(SO(n_V) \times SO(2))$	$n_V + 2$	-	
7	4	$SL(5,\mathbb{R})/SO(5)$	10	-	
	2	$\mathbb{R} \times SO(n_V,3)/(SO(n_V) \times SO(3))$	$n_V + 3$	-	
6	(2,2)	$SO(5,5)/(SO(5) \times SO(5))$	16_s	-	
	(3,1)	$F_{4(4)}/(USp(6) \times USp(2))$	(14,1)	(6,2)	no graviton
	(4,0)	$E_{6(6)}/USp(8)$	-	27	no graviton
	(3,0)	$SU^\star(6)/USp(6)$	-	14	no graviton
	(2,1)	$SU^\star(4)/USp(4)$	(4,2)	5+1	$N = (1,0)$ twin
	(2,0)	$SO(n_T,5)/(SO(n_T) \times SO(5))$	-	n_T	
	(1,1)	$\mathbb{R} \times SO(n_V,4)/(SO(n_V) \times SO(4))$	$n + 4$	-	
	(1,0)	$SO(n_T,1)/SO(n_T)$	n_V	n_T	
		$SO(9,1)/SO(9)$	16_s	9	magical
		$SO(5,1)/SO(5)$	8_s	5	magical, $N = (2,1)$ twin
		$SO(3,1)/SO(3)$	4_s	3	magical
		$SO(2,1)/SO(2)$	2_s	2	magical

Table 5 Symmetric scalar manifolds for supergravities in $D = 4, 5$ with N supersymmetry, and vector/tensor multiplet couplings. VSR and (V)SK refer to very special real and (very) special Kähler manifolds discussed in sections " $N = 2$ Supergravity Coupled to Vector, Tensor and Hypermultiplets in $5D$ and Very Special Real Manifolds" and " $N = 2$ Supergravity Coupled to Scalar and Vector Multiplets in 4D", respectively. In $5D, N = 2$ supergravity, if a subgroup $K \subset G$ can be gauged, then $n_V = \dim K$ and $n_T = n_V^{\text{tot}} - \dim K$ vectors need to be dualized to two-forms that lie in symplectic representation(s) of K , as discussed in section " $N = 2$ Supergravity Coupled to Vector, Tensor and Hypermultiplets in $5D$ and Very Special Real Manifolds". The "twins" are discussed in sections " $N = 6$ Supergravity in $5D''$ and " $N = 6$ Supergravity in $4D''$. For the r-map, see [179]

表 5 $D = 4, 5$ 维空间中具有 N 超对称，且耦合向量/张量多重态的超引力的对称标量流形。VSR 和 (V)SK 分别指代“$N = 2$ 5D 维耦合向量、张量和超多重态的超引力与非常特殊实流形”和“$N = 2$ 四维耦合标量和向量多重态的超引力”两节中讨论的非常特殊实流形和 (非常) 特殊凯勒流形。在 5D, $N = 2$ 维超引力中，若子群 $K \subset G$ 可以被规范，则 $n_V = \dim K$ 和 $n_T = n_V^{\text{tot}} - \dim K$ 个向量需要对偶化为属于 K 辛表示的二形式，相关讨论见“$N = 2$ 5D 维耦合向量、张量和超多重态的超引力与非常特殊实流形”一节。“孪生”超引力在“$N = 6$ 5D'' 维超引力”和“$N = 6$ 4D'' 维超引力”两节中讨论。关于 r-映射参见文献 [179]

D	N	Scalar Manifold G/H	n_V^{tot}	Comment
5	8	$E_6/USp(8)$	27	
	6	$SU^*(6)/USp(6)$	14+1	N=2 twin
	4	$\mathbb{R} \times SO(n_V, 5)/(SO(n_V) \times SO(5))$	$n_V + 5$	
	2	$SO(n_V, 1)/SO(n_V)$	$n_V + 1$	VSR, $n_V > 1$, nonsym. r-map image
		$SO(1, 1)$	1+1	VSR, sym. r-map image
		$\mathbb{R} \times SO(n_V - 1, 1)/SO(n_V - 1)$	$n_V + 1$	VSR
		$E_{6(-26)}/F_4$	26+1	VSR
		$SU^*(6)/Sp(3)$	14+1	VSR, N=6 twin
		$SL(3, C)/SU(3)$	8+1	VSR
		$SL(3, R)/SO(3)$	5+1	VSR
4	8	$E_{7(7)}/SU(8)$	28	
	6	$SO^\star(12)/U(6)$	16	N=2 twin
	5	$SU(5, 1)/U(5)$	10	
	4	$(SL(2, \mathbb{R}/SO(2))$		
		$\times SO(n_V, 6)/(SO(n_V) \times SO(6))$	$n_V + 6$	$n \geq 1$
	3	$SU(n_V, 3)/S[U(n_V) \times U(3)]$	$n_V + 3$	$n \geq 3$
	2	$SU(n_V, 1)/(SU(n_V) \times U(1))$	$n_V + 1$	SK, not an r-map image
		$SU(1, 1)/U(1)$	1+1	SK, r-map of pure 5D sugra
		$(SU(1, 1)/U(1))$		
		$\times SO(n_V - 1, 2)/(SO(n_V - 1) \times SO(2))$	$n_V + 1$	VSK
		$[SU(1, 1)/U(1)]^2$	2	VSK
		$E_{7(-25)}/(E_6 \times U(1))$	28	VSK
		$SO^\star(12)/U(6)$	16	VSK, N=6 twin
		$SU(3, 3)/S[U(3) \times U(3)]$	10	VSK
		$Sp(6, \mathbb{R})/U(3)$	7	VSK

Table 6 Symmetric scalar manifolds in 3D supergravities for $4 \leq N \leq 16$. The (very) special QK manifolds are discussed in section ” $(1, 0)$ Supergravity Coupled to Vector, Tensor, and Hyper Multiplets in 6D”. For ”twin” supergravities, see [165], and for the c -map, see [180]. The scalar manifold in 3D is an arbitrary Riemannian manifold for $N = 1$, Kähler for $N = 2$, quaternionic for $N = 3$, in general a product of two quaternionic manifolds for $N = 4$ and symmetric homogeneous space for $N > 4$

表 6 3D 维超引力中 $4 \leq N \leq 16$ 对应的对称标量流形。(非常) 特殊 QK 流形的讨论见“$(1, 0)$ 六维耦合向量、张量和超多重态的超引力”一节。关于“孪生”超引力参见文献 [165]，关于 c -映射参见文献 [180]。一般而言，3D 维超引力的标量流形对 $N = 1$ 是任意黎曼流形，对 $N = 2$ 是凯勒流形，对 $N = 3$ 是四元数流形，对 $N = 4$ 是两个四元数流形的直积，对 $N > 4$ 是对称齐性空间

D	N	Scalar Manifold G/H	Comment
	16	$E_{8(8)}/SO(16)$	
	12	$E_{7(-5)}/(SO(12)\times Sp(1))$	$N=4$ twin
	10	$E_{6(2)}/(SO(10)\times U(1))$	
	9	$F_{4(-20)}/SO(9)$	
	8	$SO(n,8)/(SO(n)\times SO(8))$	$N=4$ twin for $n=4$
	6	$SU(n,4)/S[U(n)\times U(4)]$	$N=4$ twin for $n=2$
	5	$Sp(n,2)/(Sp(n)\times Sp(2))$	$N=4$ twin for $n=1$
	4	$Sp(n+1,1)/(Sp(n+1)\times Sp(1))$	QK, not a c -map image, $N=5$ twin
3		$U(2,1)/(U(2)\times U(1))$	special QK, c -map of pure $4D$ sugra
		$G_{2(2)}/SO(4)$	special QK
		$SU(n+2,2)/(SU(n)\times SU(2)\times U(1))$	special QK, $N=6$ twin
		$SO(n+4,4)/(SO(n+4)\times SO(4))$	very special QK, $N=8$ twin
		$SO(3,4)/[SU(2)]^3$	very special QK
		$E_{8(-24)}/(E_7\times Sp(1))$	very special QK
		$E_{7(-5)}/(SO(12)\times Sp(1))$	very special QK, $N=12$ twin
		$E_{6(2)}/(SU(6)\times Sp(1))$	very special QK
		$F_{4(4)}/(Sp(3)\times Sp(1))$	very special QK

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